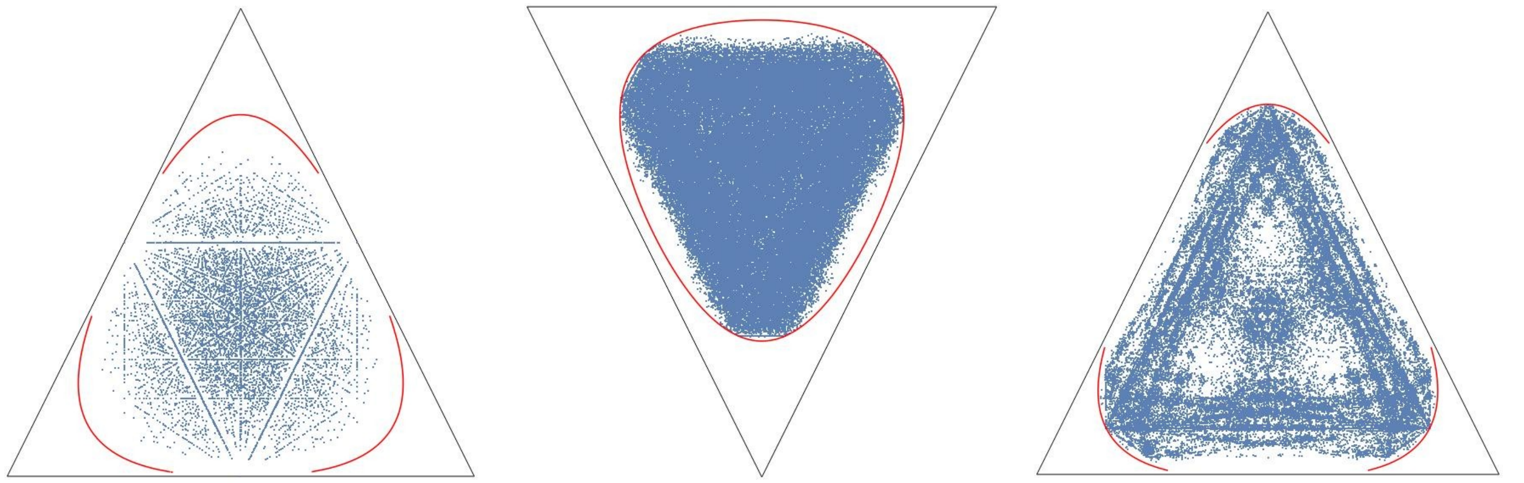


Catalytic cooling and the catalytic entropy conjecture

Henrik Wilming [ETH Zürich]

Joint work with:
Paul Boes, Jens Eisert, Rodrigo Gallego [Freie Universität Berlin]
Markus P. Müller [IQOQI Vienna, Perimeter Institute]

Phys. Rev. Lett. 122, 210402 (2019)
ArXiv:1807.08773

QTD 2019, Espoo, Finland

Von Neumann entropy

Main aim of this talk: A new way to think about von Neumann entropy

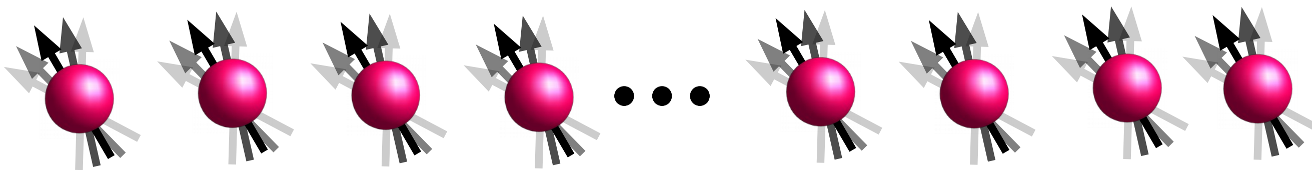
$$S(\rho) = -\mathrm{Tr}\left(\rho \log \rho\right).$$

A few results and an open problem.

Applications in thermodynamics:

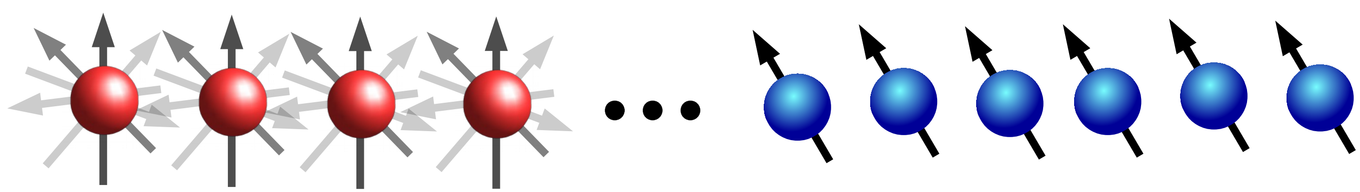
- Algorithmic cooling without *iid* limit
- Interesting ways to circumvent fluctuation relations and engineer correlated work-distributions (see **Nelly's talk** later today)

Recap: Compressing / cooling many qubits (without heat baths)



N >> 1 uncorrelated warm qubits
[each w/ **von Neumann entropy S < 1**]

Unitary U



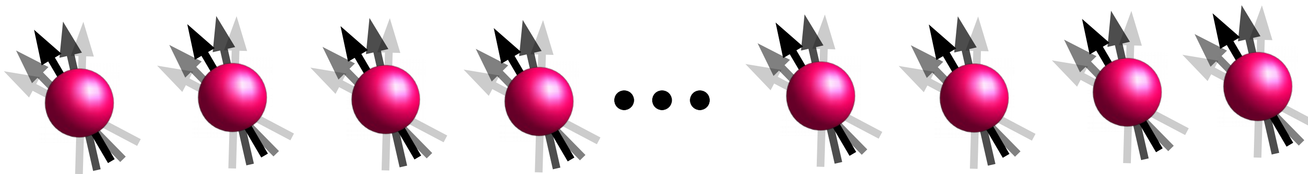
rN hot qubits
"entropy dump"

(1-r)N cold qubits

As $N \rightarrow \infty$, **r≈S** is both **necessary and sufficient** for arbitrarily good conversion.
(Abstractly: Same as compressing information).

(Shannon 1948, Schumacher 1993, Schulman & Vazirani 1999)

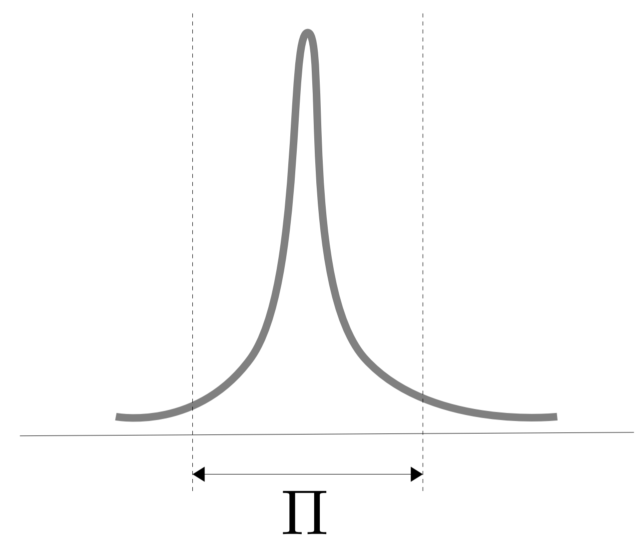
Inuition: Why von Neumann entropy?



For large n, tensor product state is highly peaked.
Well approximated by projection on **typical subspace**:

$$\rho^{\otimes n} \approx \Pi \rho^{\otimes n} \Pi$$

$$\dim(\Pi) \approx 2^{nS(\rho)}$$



We can hence **choose a basis** such that

$$\rho^{\otimes n} \approx |0\rangle\langle 0|^{\otimes (1-S)n} \otimes \sigma$$

Inuition: Why von Neumann entropy?

Similar reasoning applies in many different settings and for other standard entropic quantities:

Quantity	**Task**
Free energy	Macroscopic work extraction
Entanglement entropy	Entanglement distillation
Mutual information	Compression of correlated information sources
Relative entropy	Hypothesis testing

We always require an asymptotic *iid* limit to give operational interpretations to standard entropic quantities.

General wisdom in (quantum) information theory

Tasks in asymptotic *iid* setting:

$$\rho^{\otimes n} \longrightarrow \sigma^{\otimes m}$$

Controlled by **von Neumann entropy** (or cousins like relative entropy, mutual information etc.)

Single-shot setting:

$$\rho \longrightarrow \sigma$$

(these may be high-dimensional and consist of many correlated subsystems)

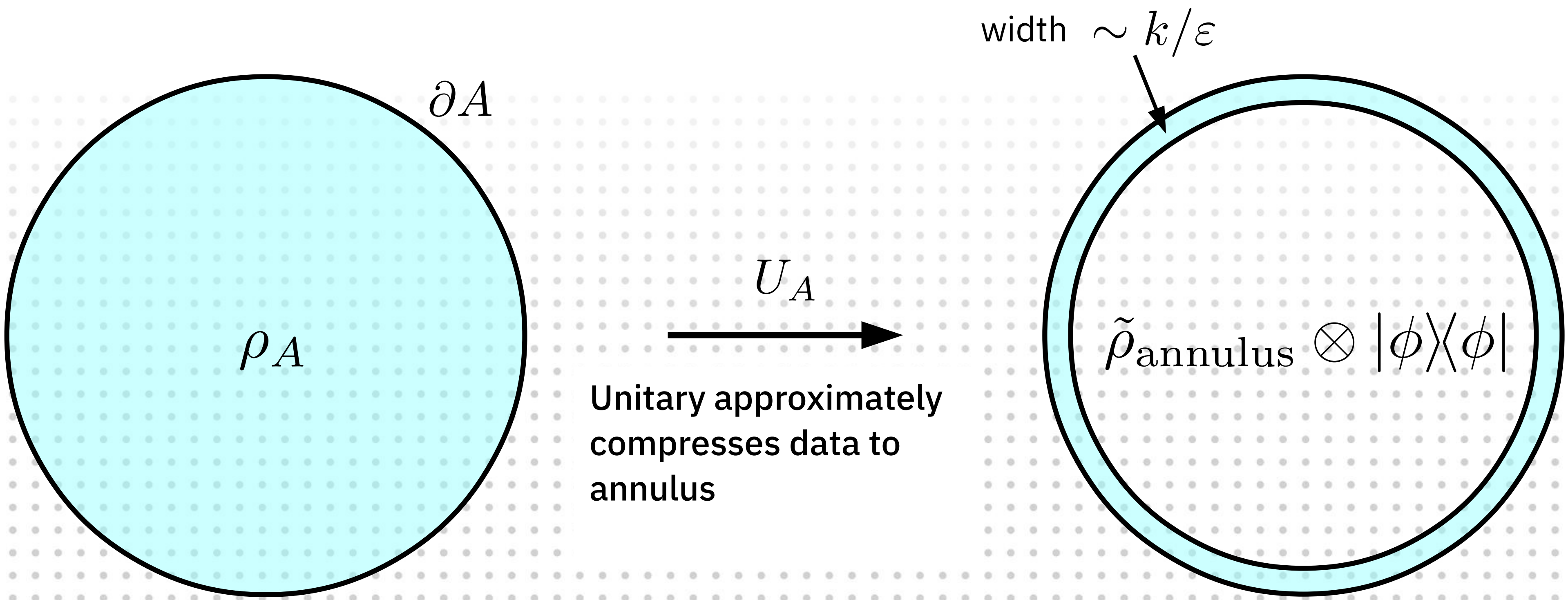
Controlled by **(smoothed) Rényi entropies** (or their relatives, s.t. single-shot free energies)

Is the *iid* limit really necessary or can we give single-shot interpretations to standard entropic quantities in certain settings?

Example outside of thermodynamics: Holographic compression from the Area Law

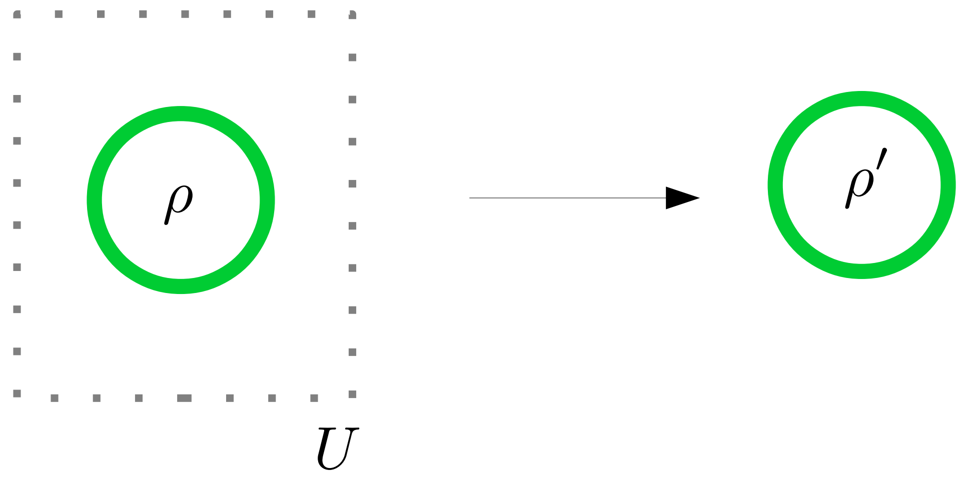
Theorem (Holographic compression, informal)

In a many-body system fulfilling an Area Law in terms of von Neumann entropy, one can unitarily compress all the information contained in a large region of space onto a thin boundary.



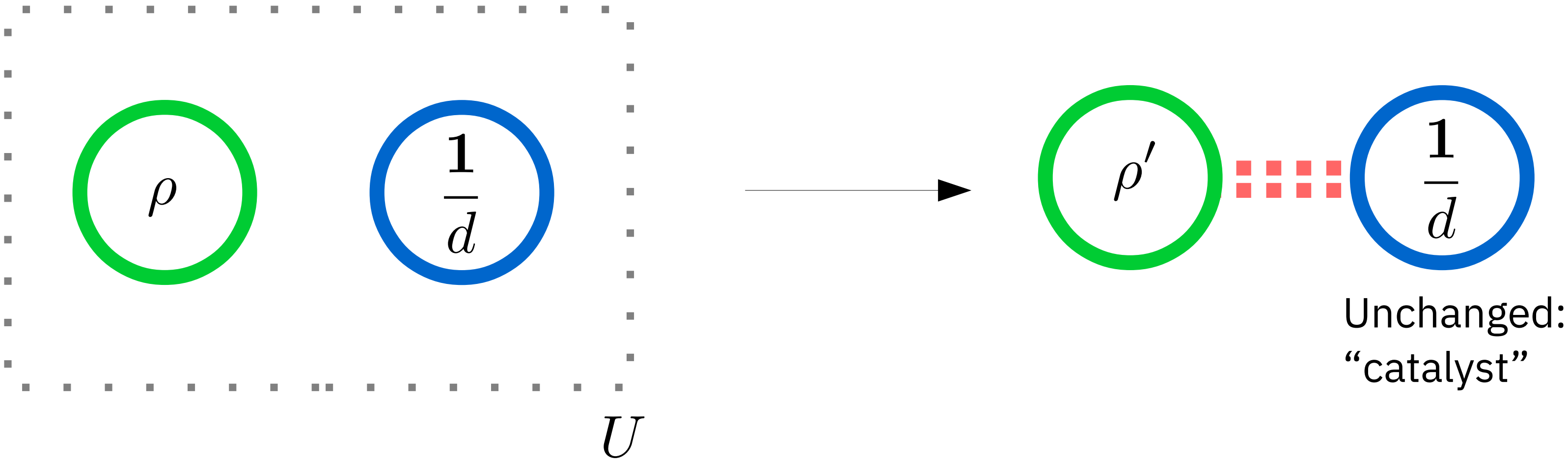
H. W. and J. Eisert
PRL 122, 190501 (2019), arXiv:1809.10156

What about general information theory and thermodynamics?



Q: Which states can be reached?

A: All states with the same spectrum.



Q: Which states can be reached by choosing different unitaries?

A: All states that are **majorized** by ρ.

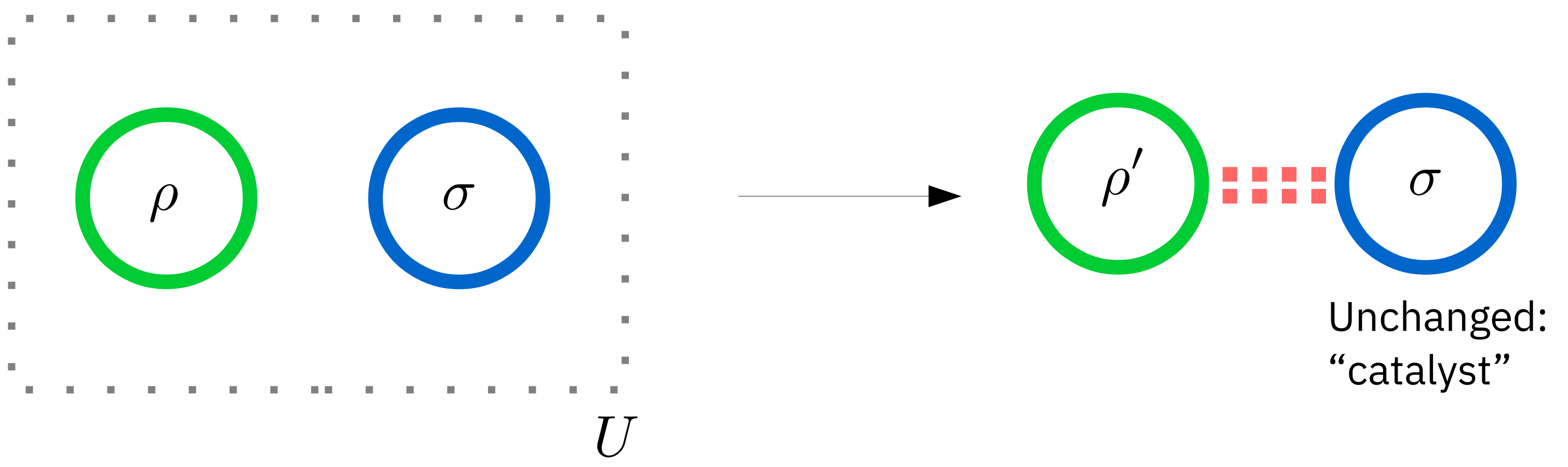
→ **All Rényi-entropies can only increase!**

Theorem

Suffices that d is larger than $\sqrt{\text{rank}(\rho')}$

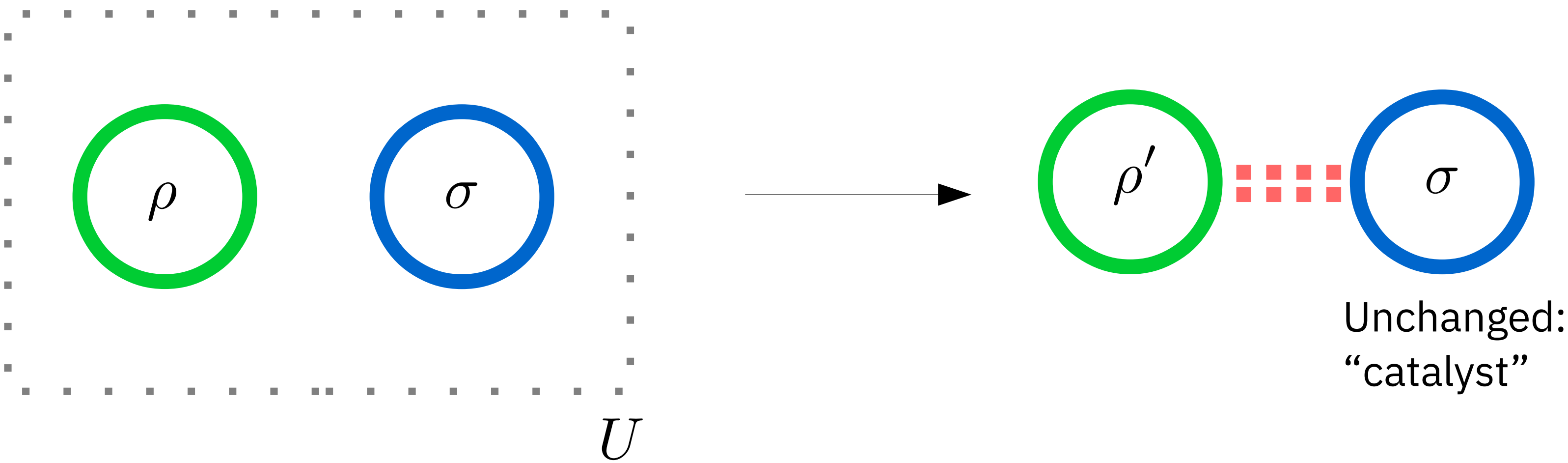
P. Boes, H.W., R. Gallego, J. Eisert, *PRX* ***8*** *(2018)*. arXiv:1804.03027

Catalytic transitions



Q: Which states can be reached if we can choose catalyst and unitary?

Catalytic transitions: Classical example



q_2p_3	q_1p_3	p_3
q_2p_2	q_1p_2	p_2
q_2p_1	q_1p_1	p_1
q_2	q_1	

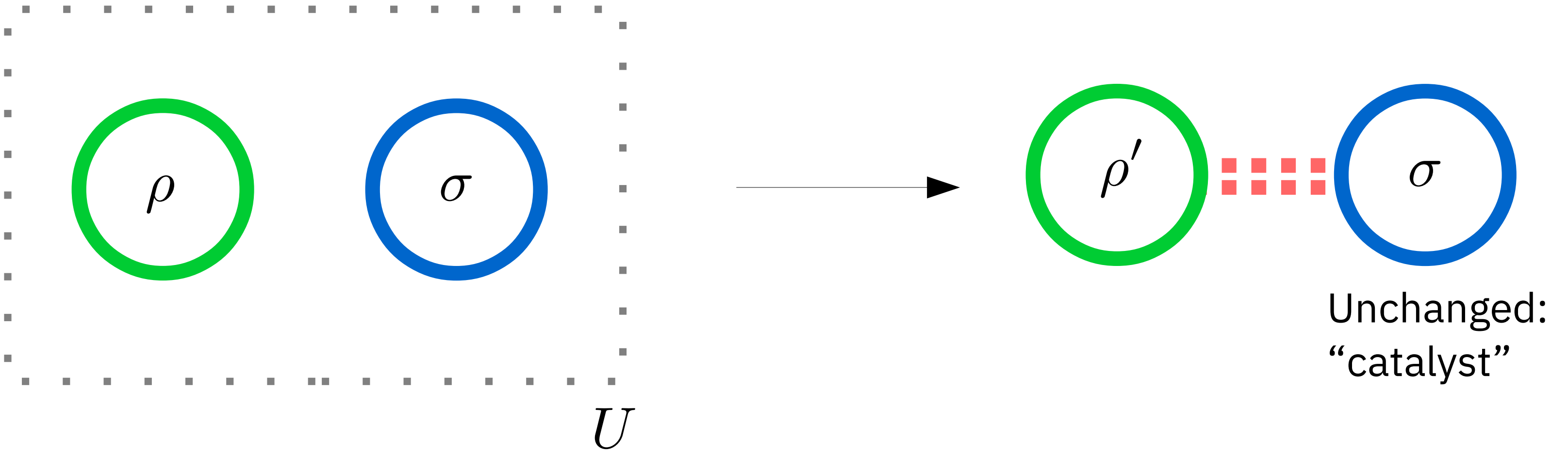
Example:
$p=(1/2,1/2,0)$

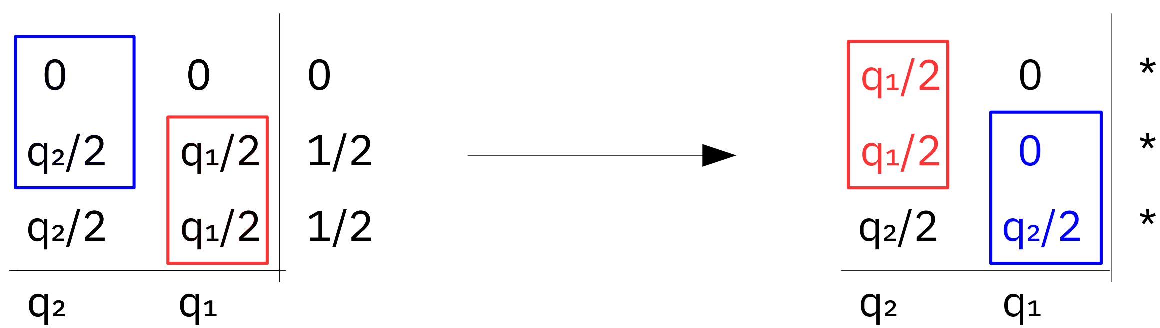
$\longrightarrow$

0	0	0
$q_2/2$	$q_1/2$	$1/2$
$q_2/2$	$q_1/2$	$1/2$
q_2	q_1	

Step 1: Write bipartite distribution as table.

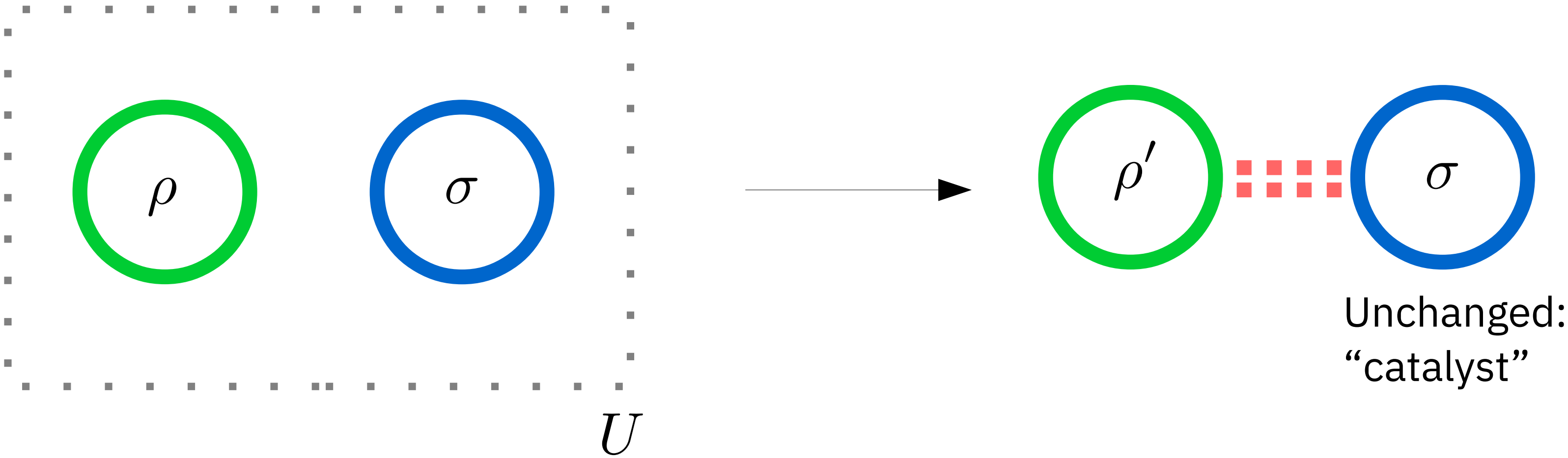
Catalytic transitions: Classical example

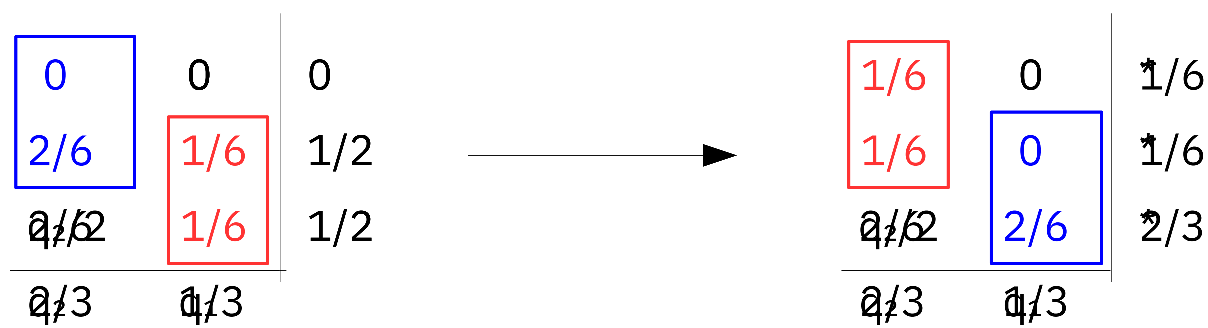




Step 2: Choose a permutation on the joint-distribution.

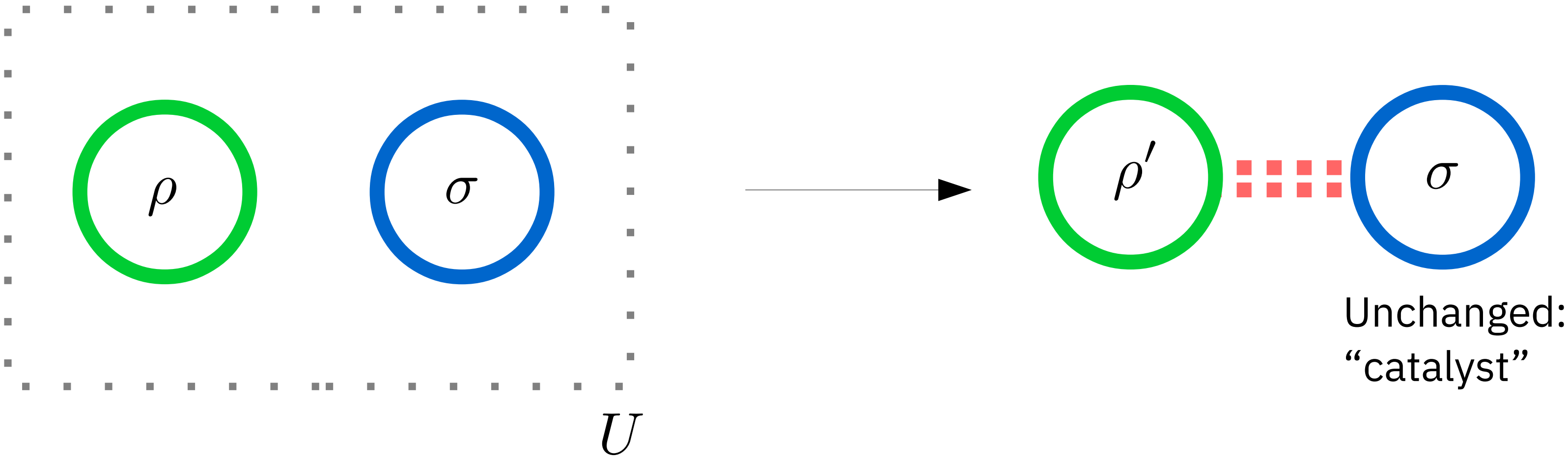
Catalytic transitions: Classical example





Step 3: Solve equations to ensure “catalyticity”.

Catalytic transitions: Classical example



0	0	0
2/6	1/6	1/2
2/6	1/6	1/2
2/3	1/3	

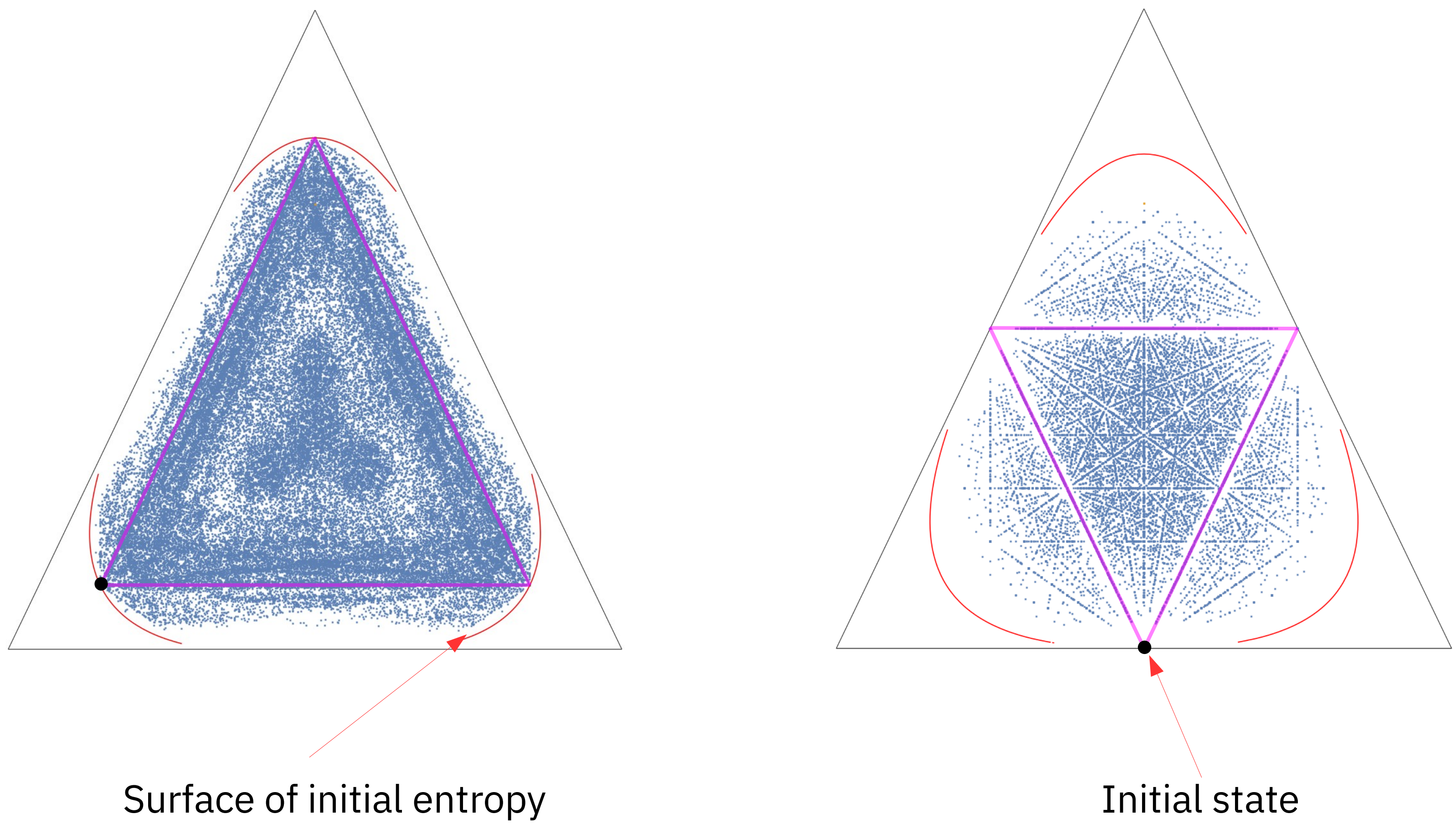
→

1/6	0	1/6
1/6	0	1/6
2/6	2/6	2/3
2/3	1/3	

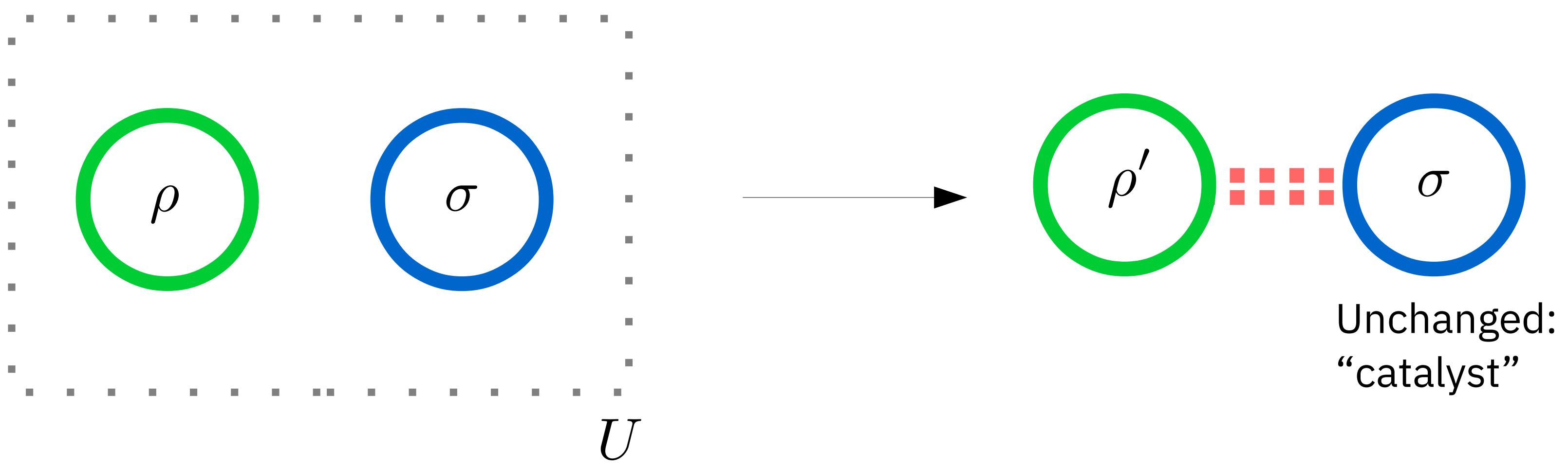
Largest probability increased from 1/2 to 2/3.

→ **Cannot be governed by majorization**

Catalytic transitions: numerics for 3-dimensional system with 5-dim catalysts



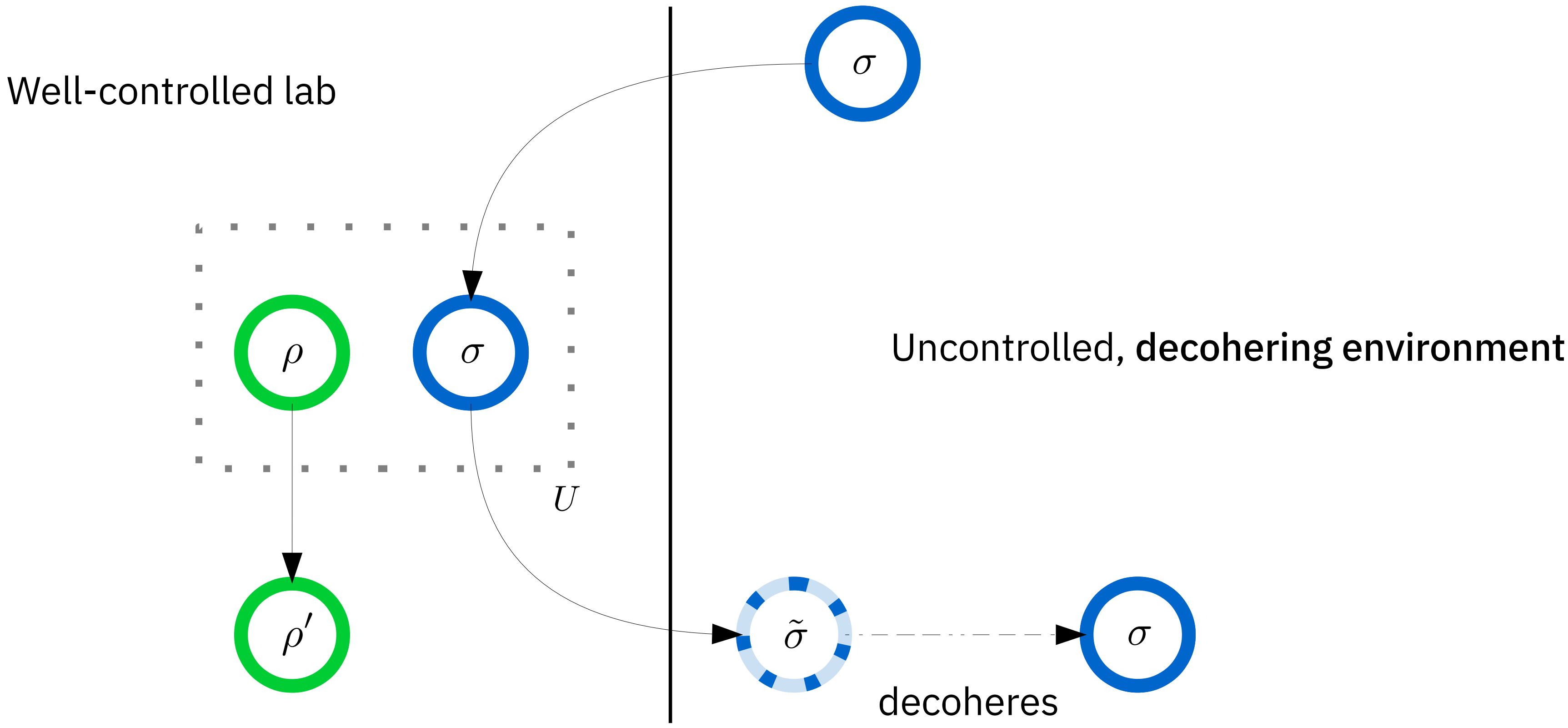
Region inside pink triangle: States reachable using **maximally mixed** catalysts ("permutahedron").



Q: Which states can be reached if we
can choose unitary and fitting catalyst?

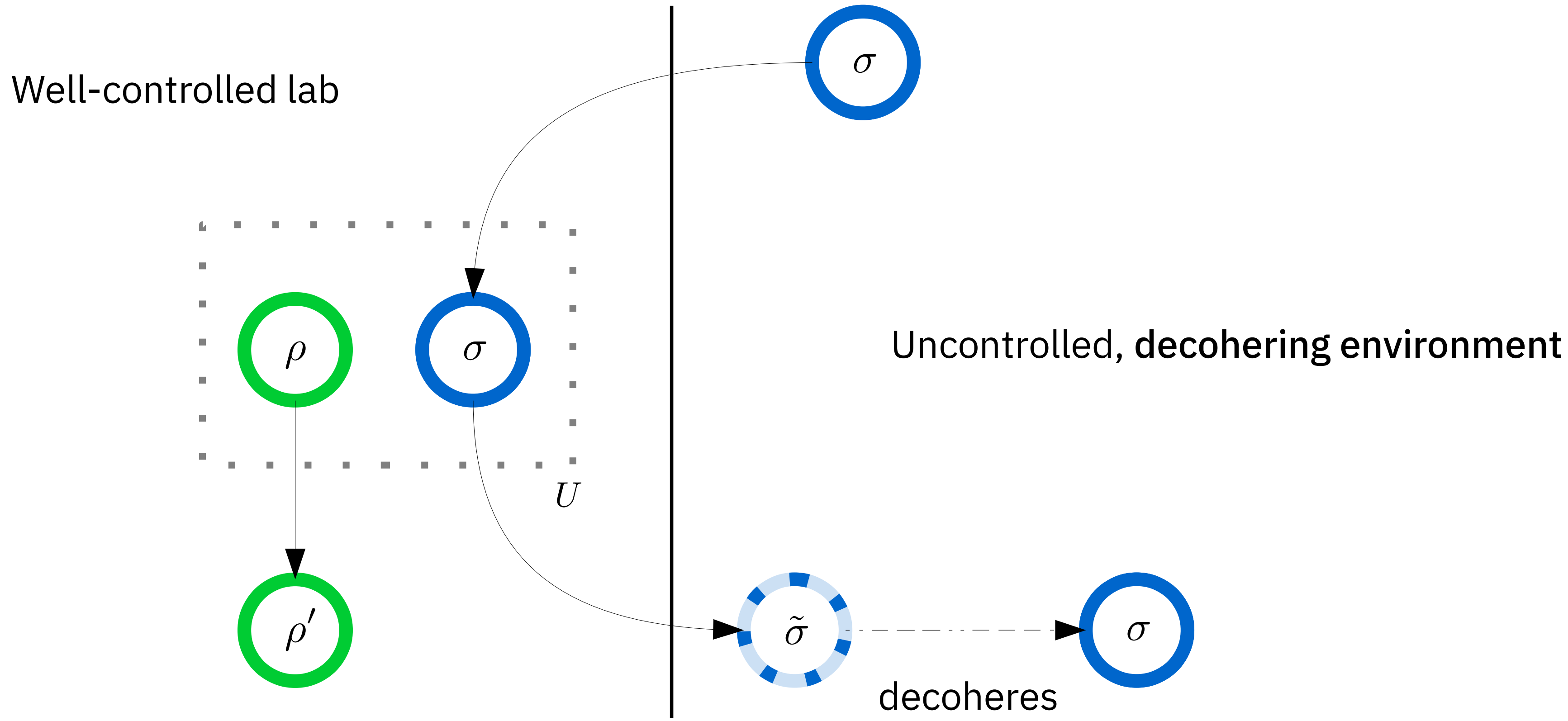
A: **Catalytic entropy conjecture?! (later)**

A more general setting



$\rho \xrightarrow{\text{dec.}} \rho' \quad :\Leftrightarrow$ Can find corresponding unitary and catalyst

A more general setting



$\rho \xrightarrow{\text{dec.}} \rho' \quad :\Leftrightarrow$

Exists unitary and catalyst such that:

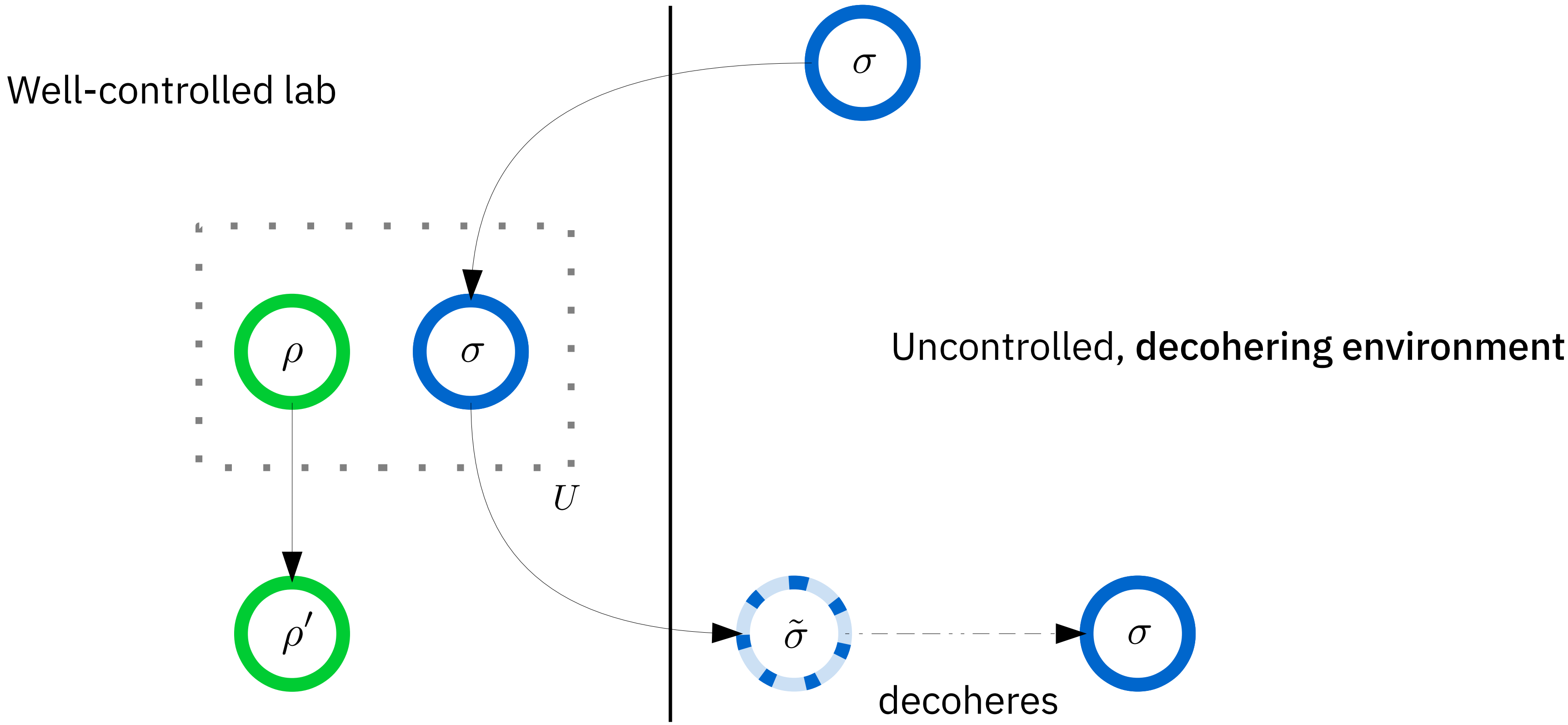
$$\mathrm{Tr}_1\left(U\rho\otimes\sigma U^\dagger\right) = \rho'$$

$$\mathcal{D}\left[\mathrm{Tr}_2\left(U\rho\otimes\sigma U^\dagger\right)\right] = \sigma$$

$\mathcal{D}$: Decoherence

Q: Which states can be reached?

A characterization of von Neumann entropy without *iid* limit



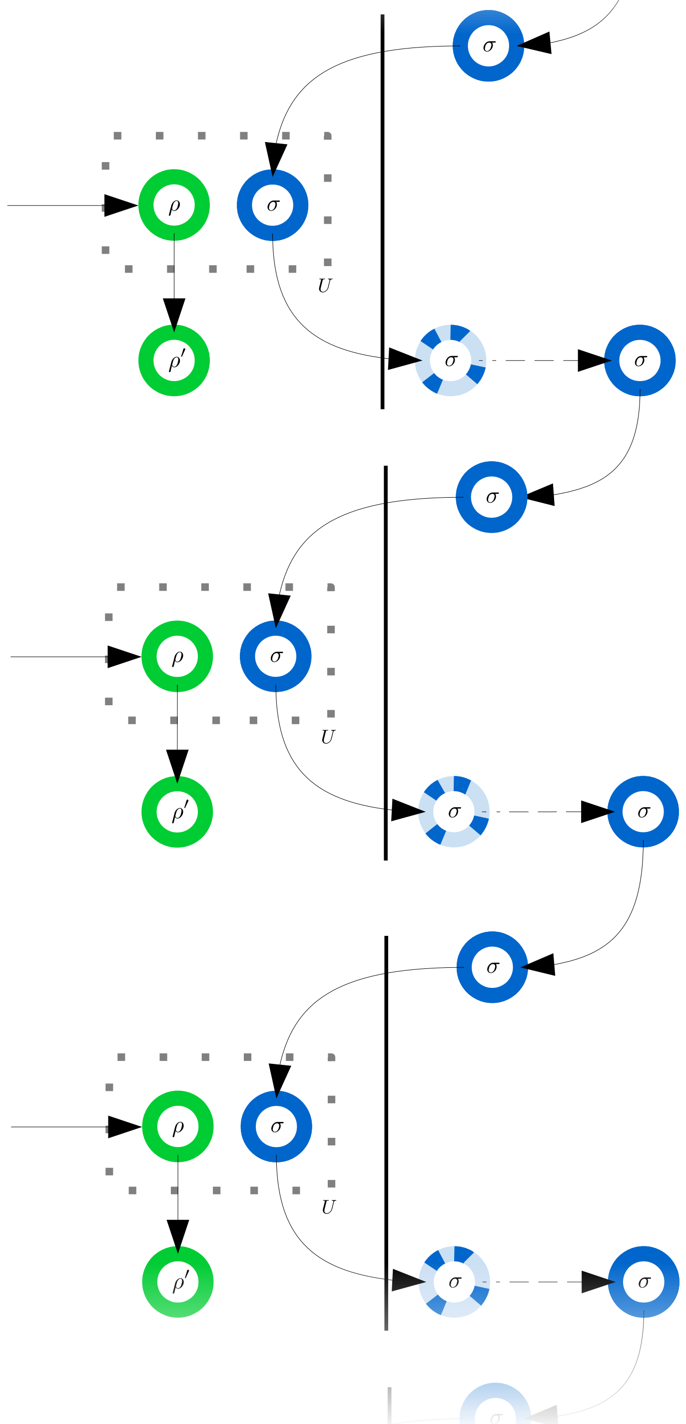
Theorem

Let $\mathrm{spec}(\rho) \neq \mathrm{spec}(\rho')$. Then:

$$\rho \xrightarrow{\text{dec.}} \rho' \quad \Leftrightarrow \quad S(\rho') > S(\rho) \quad \text{and} \quad \mathrm{rank}(\rho') \geq \mathrm{rank}(\rho)$$

Slight generalization of a result by Markus P. Müller → see his talk on Thursday!

Application: **Catalytic cooling**



Use a **single** catalyst to transform many copies. Each undergoes transition

$$\rho \longrightarrow \rho'$$

Example:

$$\rho = \chi \otimes \chi, \quad S(\chi) < \frac{1}{2}$$

$$\rho' = \frac{\mathbf{1}}{2} \otimes |0\rangle\langle 0|_\varepsilon$$

Full-rank state arbitrarily close to |0>

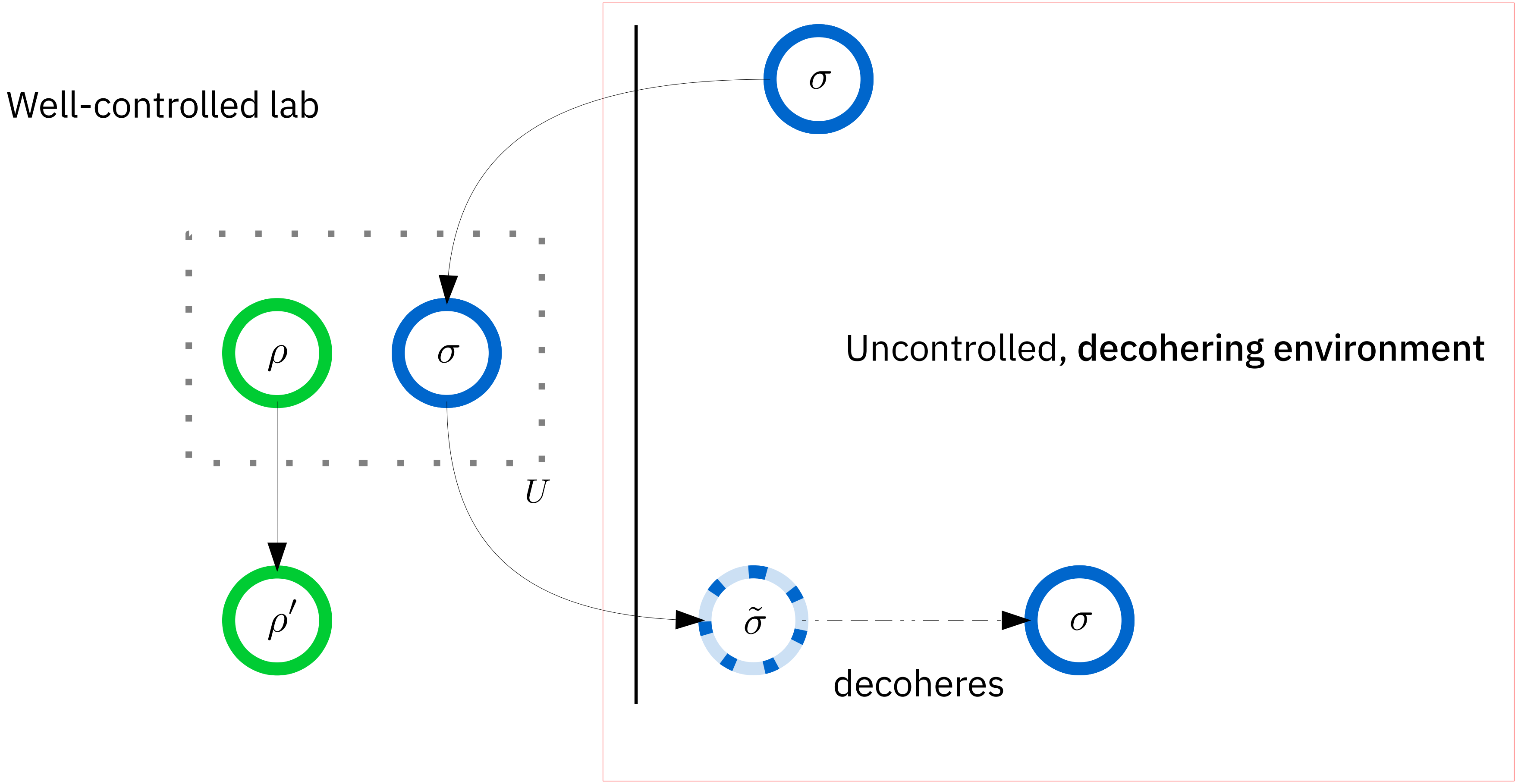
Half the systems cooled to (almost) zero temperature, half heated up to infinite temperature.

→ **Algorithmic cooling without iid limit.**

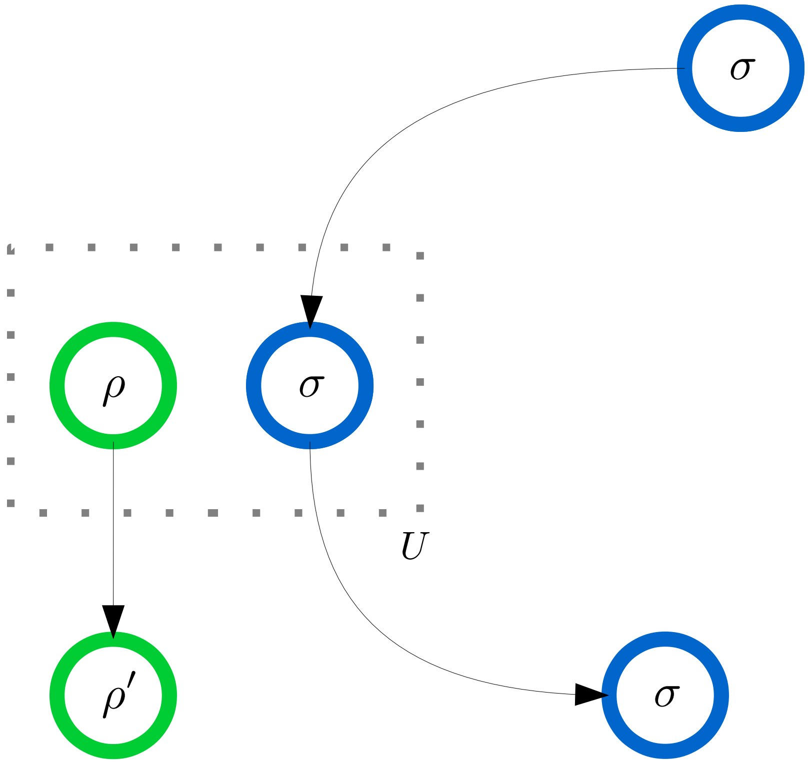
However, **correlations** are created:

$$\rho^{\otimes n} \longrightarrow \rho'_{1,\ldots,n} \neq \rho'^{\otimes n}$$

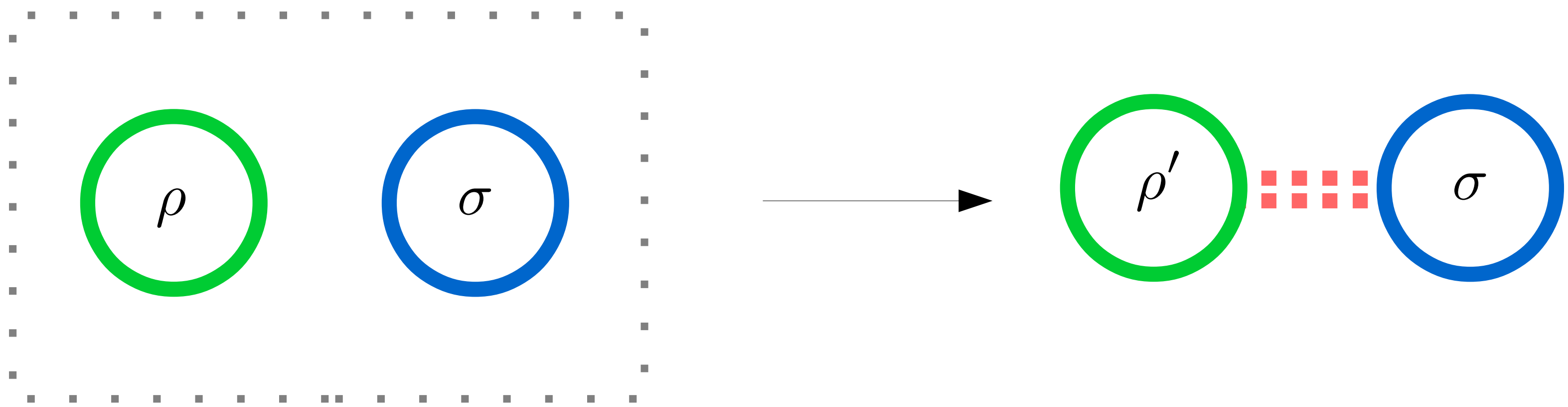
Is the decoherene necessary?



Is the decoherene necessary?

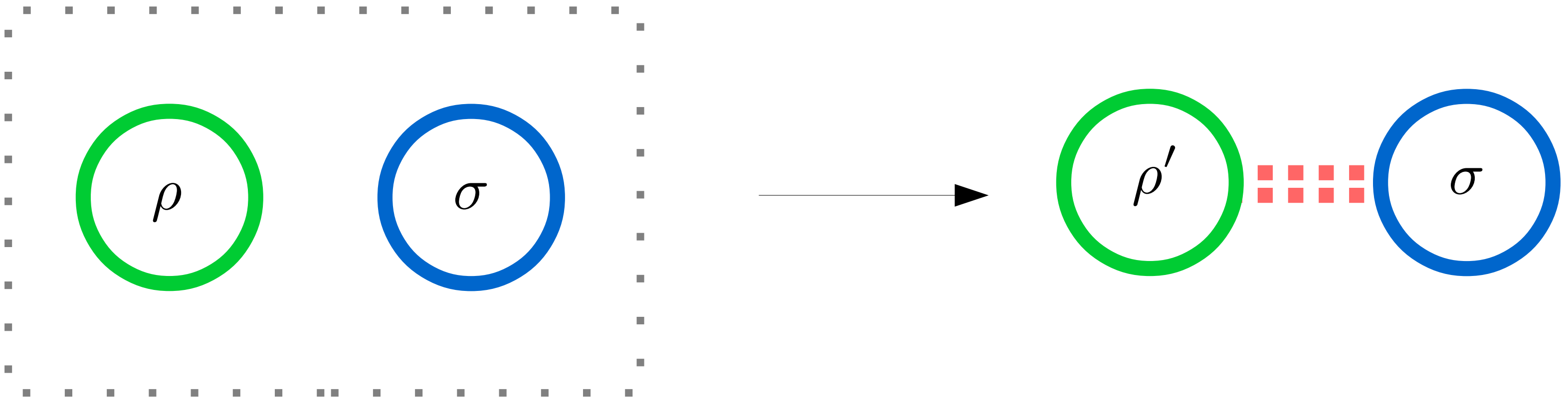


Back to catalytic transitions



Q: Which states can be reached if we can choose catalyst and unitary?

$\rho \longrightarrow \rho'$ $:\Leftrightarrow$ Can find corresponding unitary and catalyst

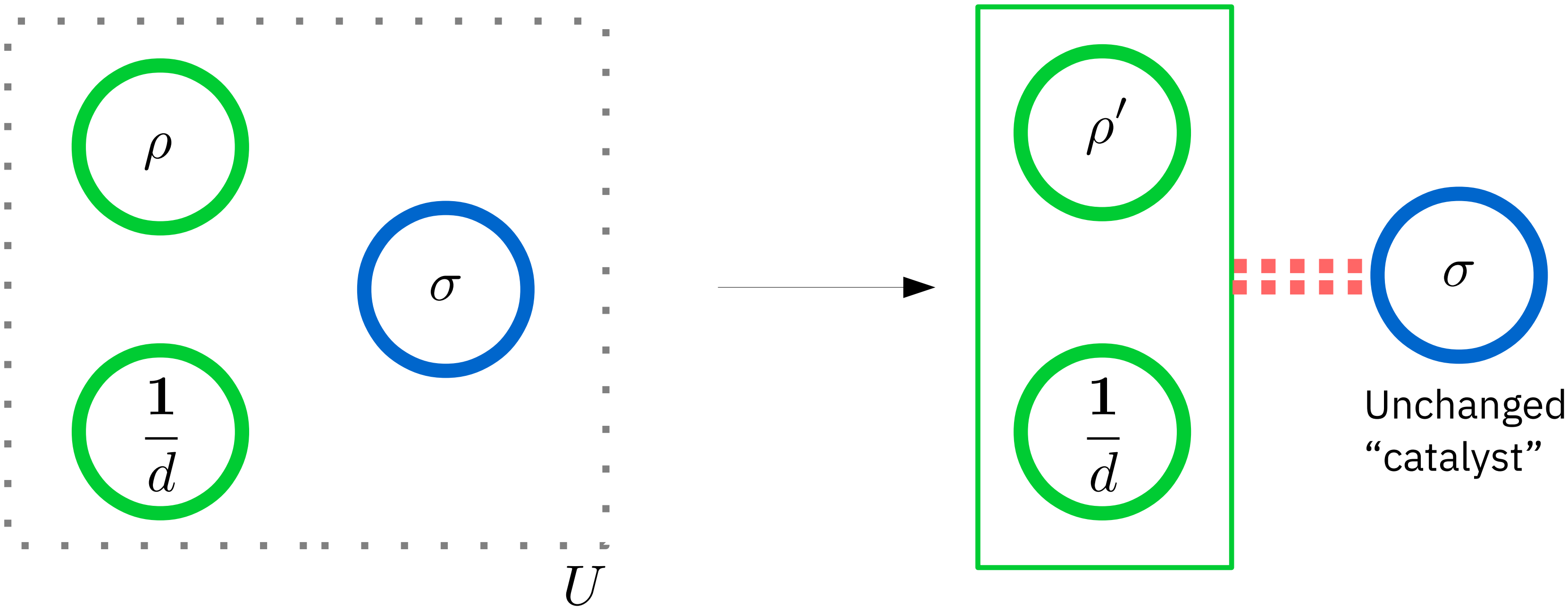


Catalytic entropy conjecture

Let $\mathrm{spec}(\rho) \neq \mathrm{spec}(\rho')$. Then:

$$\rho \longrightarrow \rho' \quad \Leftrightarrow \quad S(\rho') > S(\rho) \quad \text{and} \quad \mathrm{rank}(\rho') \geq \mathrm{rank}(\rho)$$

Weak solution to catalytic entropy conjecture



Lemma (Weak solution to conjecture)
Let $\mathrm{spec}(\rho) \neq \mathrm{spec}(\rho')$. Then the following are equivalent:

i) $S(\rho') > S(\rho)$ and $\mathrm{rank}(\rho') \geq \mathrm{rank}(\rho)$

ii) There exists some finite dimension *d* such that

$$\rho \otimes \frac{\mathbf{1}}{d} \longrightarrow \rho' \otimes \frac{\mathbf{1}}{d}$$

Implication of weak solution

Monotone: Function *f* on the set of density matrices such that

$$\rho \longrightarrow \rho' \quad \Rightarrow \quad f(\rho) \leq f(\rho')$$

Call *f* **additive** if $f(\rho_1 \otimes \rho_2) = f(\rho_1) + f(\rho_2)$.

Well-known in resource theories and other settings. Examples:

Quantity	**“Operations”**
Free energies	Thermal operations
Entanglement entropies	Local operations & classical communication (LOCC)
Asymmetry monotones	Symmetric operations
Entropy production	Markovian open system dynamics

Implication of weak solution

Monotone: Function *f* on the set of density matrices such that

$$\rho \longrightarrow \rho' \quad \Rightarrow \quad f(\rho) \leq f(\rho')$$

Call *f* **additive** if $f(\rho_1 \otimes \rho_2) = f(\rho_1) + f(\rho_2)$.

Theorem (Quasi-unique monotone, informal)
The von Neumann entropy is the **unique** (up to constants) additive and continuous monotone for catalytic transitions.

Why?
Any monotone has to be **sub-additive**:

$$f(\rho_{12}) \leq f(\rho_1) + f(\rho_2)$$

Von Neumann entropy is the only additive, continuous and sub-additive quantity (up to constants).

Summary: Characterizing von Neumann entropy through catalysts

- **Single-shot chacterization** of von Neumann entropy using catalysts & decoherence

- **Catalytic entropy conjecture**

  Let $\mathrm{spec}(\rho) \neq \mathrm{spec}(\rho')$. Then:

$$\rho \longrightarrow \rho' \quad \Leftrightarrow \quad S(\rho') > S(\rho) \quad \text{and} \quad \mathrm{rank}(\rho') \geq \mathrm{rank}(\rho)$$

Applications in thermodynamics:

- **Catalytic cooling**: Algorithmic cooling with optimal "efficiency" in the single shot

- **Circumventing Fluctuation theorems:** Using catalysts it's (in principle) possible to extract macroscopic work from macroscopic thermal systems with high probability while respecting 2$^{\text{nd}}$ Law.
  (**see talk by Nelly**)

Thank you for your attention!
Phys. Rev. Lett. 122, 210402 (2019) ArXiv:1807.08773

Holographic compression: PRL 122, 190501 (2019), arXiv:1809.10156