Design of heat pumps with parametrically driven linear electrical circuits

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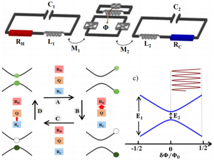
Outline

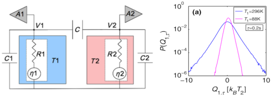
- Motivation and context. Building a general thermodynamic theory of electrical circuits.
- Quick review of the graph-theoretical description of electrical networks.
- Classical high temperature regime: when is a description of a circuit thermodynamically consistent?
- Minimal example of application: cooling a resistor
- Quantum regime: avoiding full canonical quantization.
- A fully general computational tool: Time dependent Landauer-Büttiker formula
- Quantum limits to cooling protrocols: a strong coupling effect
- Summary and next steps.

Motivation and context

- Electronic circuits are a versatile framework to experimentally explore classical and quantum non-equillibrium thermodynamics.



Karimi, B., et al., PRB 94.18 (2016): 184503



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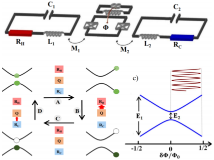

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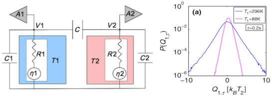

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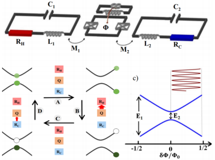

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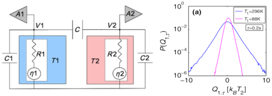

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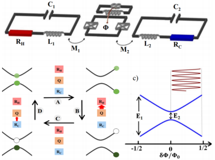

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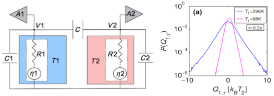

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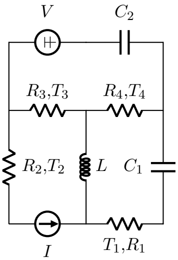
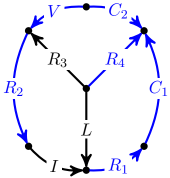
We want to build a general thermodynamic theory of electronic circuits.
First step: linear circuits.

Graph theoretical description of electrical circuits

Normal tree





Dynamical variables:

- Charges in the capacitors (q)
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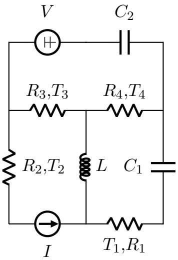

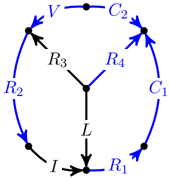

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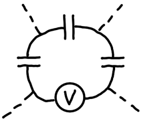
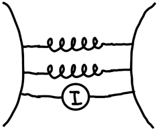
They are independent variables if there are not:



Loops of only capacitors and voltage sources



Cutsets of only inductors and current sources

Graph theoretical description of electrical circuits

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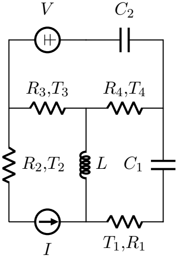

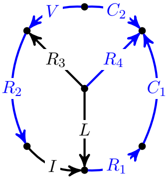

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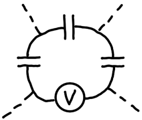

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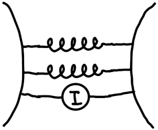


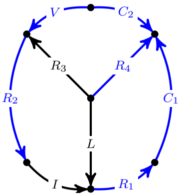
Cutsets of only inductors and current sources

Normal tree:

Under these conditions it is always possible to find a spanning tree in which all the capacitors and voltage sources are part of it, and all inductors and currents sources are out of it. Such a tree is a normal tree.

Graph theoretical description of electrical circuits

Loops and cut-sets matrices



Matrix of fundmental loops:

Adding edges to the tree one by one we can form loops...

$$
\begin{array}{c|cccccc}
 & V & C_1 & C_2 & R_1 & R_2 & R_4 \\
R_3 & -1 & 0 & 1 & 0 & 0 & -1 \\
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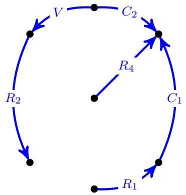

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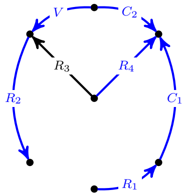

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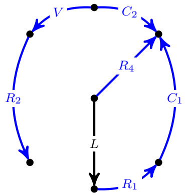

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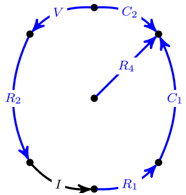

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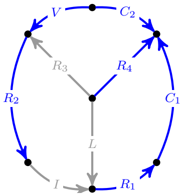

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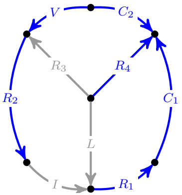

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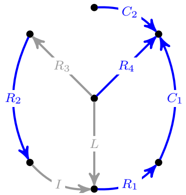

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Graph theoretical description of electrical circuits

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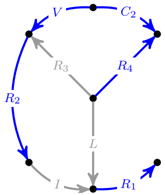

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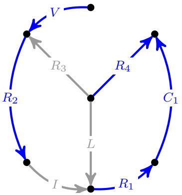

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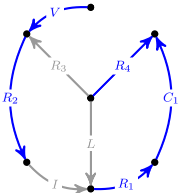
Graph theoretical description of electrical circuits

Loops and cut-sets matrices



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These descriptions are equivalent: $Q_{\text{link}} = -B_{\text{twig}}^T$

Graph theoretical description of electrical circuits

Cut-sets matrix and state equation

Thus, given a normal tree we can construct a set of fundamental cut-sets:

$$Q_{\mathsf{link}} = \begin{bmatrix} Q_{\mathsf{ER}} & Q_{\mathsf{EL}} & Q_{\mathsf{EI}} \\ Q_{\mathsf{CR}} & Q_{\mathsf{CL}} & Q_{\mathsf{CI}} \\ Q_{\mathsf{RR}} & Q_{\mathsf{RL}} & Q_{\mathsf{RI}} \end{bmatrix}$$

and from its blocks we can describe the dynamics of the circuit state:

$$x = \begin{bmatrix} q \\ \phi \end{bmatrix} \quad s = \begin{bmatrix} v_E \\ j_I \end{bmatrix} \qquad \boxed{\frac{dx}{dt} = \mathcal{A}(t)\mathcal{H}(t)\, x + \mathcal{B}(t)s(t)} \qquad \mathcal{H} = \begin{bmatrix} C^{-1} & \\ & L^{-1} \end{bmatrix}$$

$$\mathcal{A}(t) = \mathcal{M}_c - \mathcal{M}_d^T \alpha \mathcal{M}_d$$

$$\mathcal{B}(t) = \mathcal{M}_s - \mathcal{M}_d^T \alpha \mathcal{M}_{sd}$$

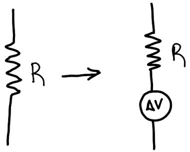
$$\mathcal{M}_c = \begin{bmatrix} & -Q_{\mathsf{CL}} \\ Q_{\mathsf{CL}}^T & \end{bmatrix}$$

$$\mathcal{M}_d = \begin{bmatrix} Q_{\mathsf{CR}}^T & \\ & -Q_{\mathsf{RL}} \end{bmatrix}$$

$$\alpha = \begin{bmatrix} R_l & -Q_{\mathsf{RR}}^T \\ Q_{\mathsf{RR}} & R_t^{-1} \end{bmatrix}^{-1}$$

Stochastic dynamics

High temperature Johnson-Nyquist noise:



- $\langle \Delta v(t) \rangle = 0$
- $\langle \Delta v(t) \Delta v(t') \rangle = 2 R k_b T \, \delta(t - t')$

Langevin state equation

$$\boxed{\frac{dx}{dt} = \mathcal{A}(t)\mathcal{H}(t)\, x + \mathcal{B}(t) s(t) + \sum_r \sqrt{2 k_b T_r}\, \mathcal{C}_r \, \xi(t)}$$

$$\mathcal{C}_r = \mathcal{M}_d^T \alpha R^{1/2} \Pi_r \qquad \langle \xi_i(t) \xi_j(t') \rangle = \delta_{i,j} \delta(t - t')$$

Fluctuation dissipation relation:

$$(\mathcal{A})_s = \frac{\mathcal{A} + \mathcal{A}^T}{2} = -\sum_r \mathcal{C}_r \mathcal{C}_r^T$$

Stochastic dynamics

Evolution of the mean values and covariance matrix

Mean values evolve with the deterministic equation of motion:

$$\frac{d\langle x\rangle}{dt} = \mathcal{A}\mathcal{H}(t)\,\langle x\rangle + \mathcal{B}(t)s(t)$$

And the covariance matrix $\sigma = \langle xx^T\rangle - \langle x\rangle\langle x\rangle^T$ evolves acording to:

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_bT_r\,\mathcal{C}_r\mathcal{C}_r^T$$

For time independent systems the stationary state covariance matrix can be obtained by solving a Lyapunov equation:

$$0 = \mathcal{A}\mathcal{H}\sigma_{\mathsf{st}} + \sigma_{\mathsf{st}}\mathcal{H}\mathcal{A}^T + \sum_r 2k_bT_r\,\mathcal{C}_r\mathcal{C}_r^T$$

Stochastic thermodynamics

Definition of local heat currents

Circuit energy and its variation:

$$E = \frac{1}{2} x^T \mathcal{H}(t) x \qquad \Longrightarrow \qquad \langle E \rangle = \frac{1}{2} \operatorname{Tr}\left[\mathcal{H}(t) \langle x \rangle \langle x \rangle^T\right] + \frac{1}{2} \operatorname{Tr}\left[\mathcal{H}\sigma\right]$$

$$\frac{d\langle E \rangle}{dt} = \underbrace{\frac{1}{2} \operatorname{Tr}\left[\mathcal{H}(t) \frac{d}{dt}\left(\langle x \rangle \langle x \rangle^T + \sigma\right)\right]}_{\text{Heat}} + \underbrace{\frac{1}{2} \operatorname{Tr}\left[\frac{d}{dt}\mathcal{H}(t)\left(\langle x \rangle \langle x \rangle^T + \sigma\right)\right]}_{\text{Work}}$$

Employing the evolution equation for σ and the FD relation, we obtain:

$$\langle \dot{Q} \rangle = \sum_r \left(\langle j_r \rangle \langle v_r \rangle + \operatorname{Tr}[(\mathcal{H}\sigma\mathcal{H} - k_b T_r \mathcal{H}) \mathcal{C}_r \mathcal{C}_r^T]\right),$$

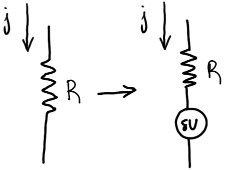
From this it is natural to define the <u>local heat currents</u> as:

$$\langle \dot{Q}_r \rangle = \langle j_r \rangle \langle v_r \rangle + \operatorname{Tr}[(\mathcal{H}\sigma\mathcal{H} - k_b T_r \mathcal{H}) \mathcal{C}_r \mathcal{C}_r^T].$$

However, <u>THIS IS NOT ALWAYS CORRECT!</u>

Stochastic thermodynamics

Definition of local heat currents

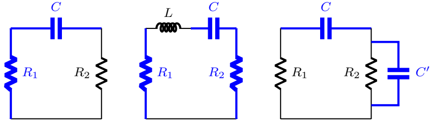


Local heat currents can be naturally defined as:

$$\dot{Q}_r = j_r(v_r + \Delta v_r)$$

However, $\langle \dot{Q}_r \rangle$ is divergent in general. Why?

<u>Some examples:</u>



(a) (b) (c)

In (a), fluctuations of arbitrarily high frequency in R_2 can be dissipated into R_1.
In (b) and (c) these fluctuations are filtered out.

Stochastic thermodynamics

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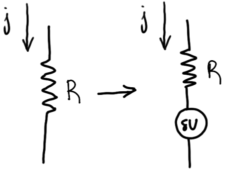

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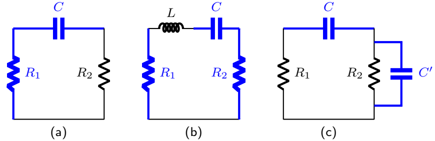

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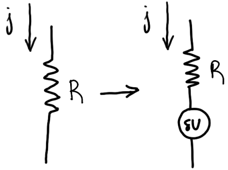
Stochastic thermodynamics

Definition of local heat currents



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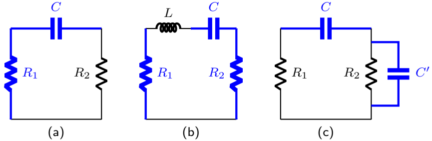

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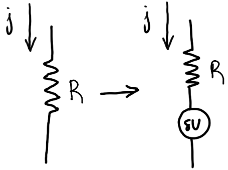

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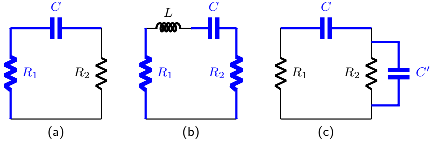

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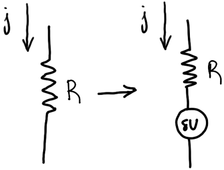

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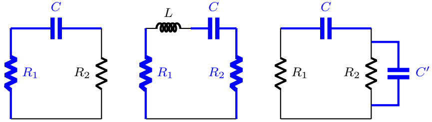

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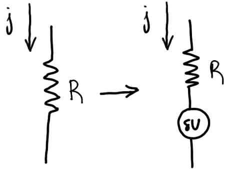

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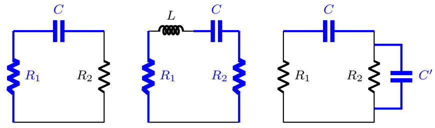

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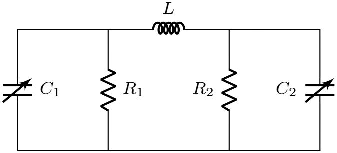
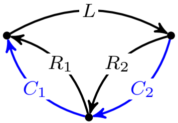
- Or, equivalently, by 'dressing' a white noise resistor (analogous to Markovian embedding techniques).
- General topological condition:
  All heat currents are well defined if there are no fundamental cut-sets simultaneously involving resistors inside and outside the normal tree.

$$Q_{\mathrm{RR}} = 0$$

Application: cooling a resistor

A simple circuit-based machine





The matrices describing this circuit are:

$$Q_{\text{link}} = \Big[\underbrace{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}_{Q_{\text{CR}}} \Big| \underbrace{\begin{array}{c} -1 \\ -1 \end{array}}_{Q_{\text{CL}}}\Big] \implies \mathcal{M}_c = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \hline -1 & -1 & 0 \end{array}\right] \quad \mathcal{M}_d = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

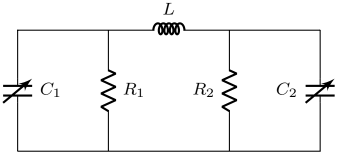
And the state equation is:

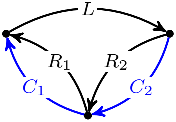
$$\frac{dx}{dt} = \mathcal{A}(t)\mathcal{H}(t)\, x + \sqrt{2k_b T_1}\, \mathcal{C}_1\, \xi(t) + \sqrt{2k_b T_2}\, \mathcal{C}_2\, \xi(t)$$

$$\mathcal{A}(t) = \mathcal{M}_c - \mathcal{M}_d^T \alpha \mathcal{M}_d = \left[\begin{array}{cc|c} -R_1^{-1} & 0 & 1 \\ 0 & -R_2^{-1} & 1 \\ \hline -1 & -1 & 0 \end{array}\right]$$

Application: cooling a resistor

A simple circuit-based machine





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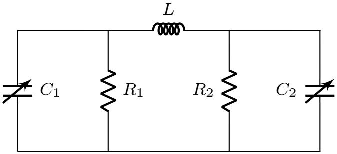
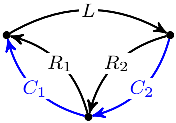
$$\mathcal{H}(t) = \begin{bmatrix} C_1(t) & & \\ & C_2(t) & \\ & & L \end{bmatrix}$$

Application: cooling a resistor

A simple circuit-based machine





1 - Stationary heat conduction: $(C_1 = C_2 = C,\ R_1 = R_2 = R)$

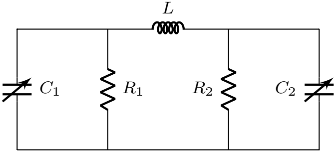
$$0 = \mathcal{A}\mathcal{H}\sigma_{\mathsf{st}} + \sigma_{\mathsf{st}}\mathcal{H}\mathcal{A}^T + \sum_r 2k_b T_r\, \mathcal{C}_r\mathcal{C}_r^T$$

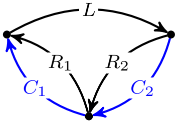
$$\sigma_{\mathsf{st}} = k_b\bar{T}\mathcal{H}^{-1} + \frac{k_b\Delta T}{2}\frac{CL}{CR^2+L}\begin{bmatrix} 1 & 0 & -R \\ 0 & -1 & R \\ -R & R & 0 \end{bmatrix}$$

$$\langle\dot{Q}_r\rangle = \mathrm{Tr}[(\mathcal{H}\sigma\mathcal{H} - k_bT_r\mathcal{H})\mathcal{C}_r\mathcal{C}_r^T] \quad\Longrightarrow\quad \langle\dot{Q}_1\rangle = -\langle\dot{Q}_2\rangle = -\frac{k_b\Delta T}{2}\frac{R}{CR^2+L}$$

Application: cooling a resistor

A simple circuit-based machine





2 - Isothermal refrigeration ($T_1 = T_2 = T$)

We consider a simple driving protocol:

$$C_1 = C + \Delta C \cos(\omega_d t)$$
$$C_2 = C + \Delta C \cos(\omega_d t + \phi)$$

And compute the asymptotic cycle averages:

$$\langle \dot{X} \rangle_c = \lim_{t\to\infty} \frac{\omega_d}{2\pi} \int_t^{t+\frac{2\pi}{\omega_d}} \langle \dot{X} \rangle$$

In this case, we need to solve:

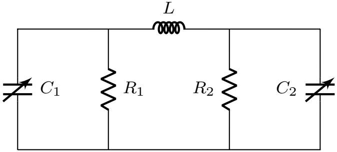
$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r\, \mathcal{C}_r \mathcal{C}_r^T$$

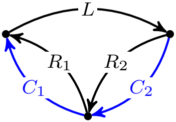
For periodic driving, under stability conditions, $\sigma(t)$ is asymptotically periodic:

$$\mathcal{H}(t) = \sum_{k=-\infty}^{+\infty} \mathcal{H}_k\, e^{ik\omega_d t}. \quad \underset{\text{asymptotically}}{\Longrightarrow} \quad \sigma(t) = \sum_{k,k'=-\infty}^{+\infty} \sigma_{k,k'}\, e^{i(k-k')\omega_d t}$$

Application: cooling a resistor

A simple circuit-based machine





3 - Isothermal refrigeration $(T_1 = T_2 = T)$

In the weak driving $(\Delta C \ll C)$ and adiabatic $(\omega_d \ll 1/\sqrt{LC})$ limits we can obtain analytical results:

$$\langle \dot{W} \rangle_c = \langle \dot{Q}_1 \rangle_c + \langle \dot{Q}_2 \rangle_c = k_b T \, (\omega_d \Delta C)^2 \, \frac{R(CR^2 \cos(\theta) + CR^2 + 2L)}{8C(CR^2 + L)} + \mathcal{O}(\omega_d^3)$$

$$\langle \dot{Q}_{1/2} \rangle_c = \mp k_b T \, \omega_d (\Delta C)^2 \frac{R^4 \sin(\theta)}{8(CR^2 + L)^2} + \frac{\langle \dot{W} \rangle_c}{2} + \mathcal{O}(\omega_d^3)$$

Coefficient of Perfomance:

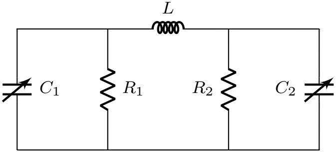
$$\text{CoP} = \frac{|\langle \dot{Q}_1 \rangle_c|}{\langle \dot{W} \rangle_c} = \frac{1}{\omega_d} \frac{R \sin(\theta)/(CR^2 + L)}{\cos(\theta) + 1 + 2L/(CR^2)} - \frac{1}{2}$$

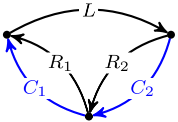
Maximum driving frequency:

$$\omega_d^{\text{max}} = \frac{2(RC)^{-1}}{1 + L/(CR^2)} \frac{\sin(\theta)}{\cos(\theta) + 1 + 2L/(CR^2)}$$

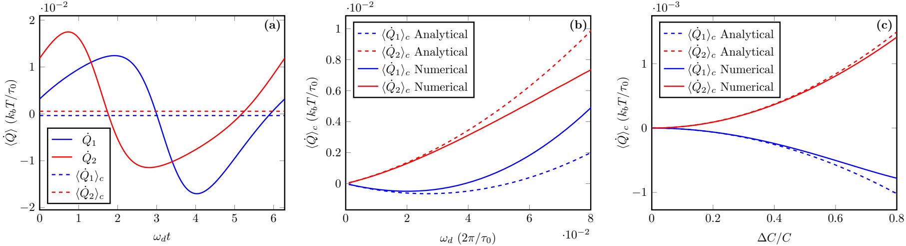
Application: cooling a resistor

A simple circuit-based machine





Numerical vs analytical results: ($\tau_0 = \sqrt{LC}$, $\tau_d = RC$, $\tau_0 = \tau_d$)



(a) Asymptotic cycle of the heat currents for $\Delta C/C = 1/2$ and $\omega_d/(2\pi) = 10^{-2}/\tau_d$ (dashed lines indicate cycle averages).
(b) Average heat currents versus driving frequency for $\Delta C/C = 0.5$.
(c) Average heat currents versus driving strength for $\omega_d/(2\pi) = 10^{-2}/\tau_d$.
For all cases we took $\theta = \pi/2$ and $T_1 = T_2 = T$.

How were these results obtained?

Generalized Lyapunov equation

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r\, \mathcal{C}_r\mathcal{C}_r^T$$

$$\mathcal{H}(t) = \sum_{k=-\infty}^{+\infty} \mathcal{H}_k\, e^{ik\omega_d t}. \quad \underset{\text{asymptotically}}{\Longrightarrow} \quad \sigma(t) = \sum_{k,k'=-\infty}^{+\infty} \sigma_{k,k'}\, e^{i(k-k')\omega_d t}$$

How to find the $\sigma_{k,k'}$ given $\mathcal{H}_k$? We can define:

$$S = \begin{bmatrix} \ddots & & & & \\ & \sigma^2_{-1,-1} & \sigma^2_{-1,0} & \sigma^2_{-1,1} & \\ & \sigma^2_{0,-1} & \sigma^2_{0,0} & \sigma^2_{0,1} & \\ & \sigma^2_{1,-1} & \sigma^2_{1,0} & \sigma^2_{1,1} & \\ & & & & \ddots \end{bmatrix} \qquad D_r = \begin{bmatrix} \ddots & & & & \\ & 0 & 0 & 0 & \\ & 0 & \mathcal{C}_r\mathcal{C}_r^T & 0 & \\ & 0 & 0 & 0 & \\ & & & & \ddots \end{bmatrix}$$

And then just solve the Lyapunov equation:

$$AS + SA^\dagger + \sum_r 2k_b T_r D_r = 0$$

This is a particularly simple problem well suited for automatic optimization.

Generalization to quantum noise

Preliminary comments

Canonical quantization of electrical circuits:

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Generalization to quantum noise

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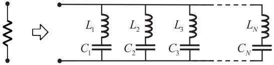
4. If there are resistors, we can explicitly model them as collections of harmonic modes (Caldeira-Legget model)



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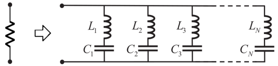

5. In the end, we obtain:

$$H_T = H_{\text{circuit}} + \sum_r H_r + \sum_r H_{\text{int},r}$$

Generalization to quantum noise

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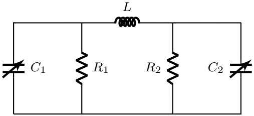
There is a limitation: it requires a full specification of stray degrees of freedom



This circuit cannot be directly quantized

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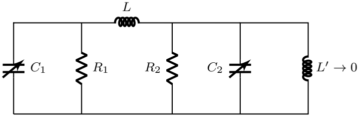

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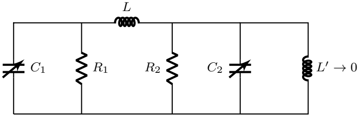

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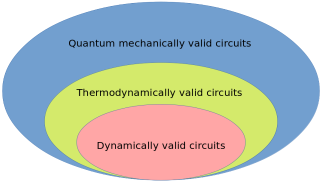


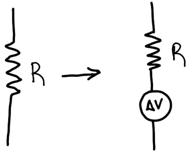
We would like to have a tool to study circuits at low temperatures without having to worry about irrelevant degrees of freedom.

Generalization to quantum noise

Semiclassical treatment

Classical Johnson-Nyquist noise:

$$\langle \Delta v(t) \Delta v(t') \rangle = 2Rk_bT\, \delta(t-t') \implies S(\omega) = \frac{Rk_bT}{\pi}$$



Quantum Johnshon-Nyquist noise:

$$`\langle \Delta v(t) \Delta v(t') \rangle ' = \left\langle \frac{\Delta v(t) \Delta v(t') + \Delta v(t') \Delta v(t)}{2} \right\rangle = f(t,t')$$

$$\begin{aligned} \mathcal{S}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau\, e^{-i\omega\tau} f(t,t') &= \frac{R}{\pi} \hbar\omega\ \coth\left(\frac{\hbar\omega}{2k_bT}\right) \\ &= \frac{R}{2\pi} \hbar\omega\, (N(\omega) + 1/2) \end{aligned}$$

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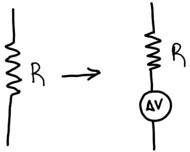

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$$\mathcal{S}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau \, e^{-i\omega\tau} f(t, t') = \frac{R}{\pi} \hbar\omega \, \coth\left(\frac{\hbar\omega}{2 k_b T} \right)$$

$$= \frac{R}{2\pi} \hbar\omega \, (N(\omega) + 1/2)$$

Semiclassical treatment:

$$\frac{dx}{dt} = \mathcal{A}(t) \mathcal{H}(t) \, x + \mathcal{B}(t) s(t) + \sum_r \sqrt{2 k_b T_r} \, \mathcal{C}_r \, \xi(t), \qquad \mathcal{S}_{\xi_r}(\omega) = \frac{1}{2\pi} \frac{\hbar\omega}{k_b T_r} \, (N_r(\omega) + 1/2)$$

We do not promote x to quantum operators!

Generalization to quantum noise

Green's function techniques

For $y = x - \langle x \rangle$

$$\frac{dy}{dt} = \mathcal{A}(t)\mathcal{H}(t)\, y + \sum_r \sqrt{2k_b T_r}\, \mathcal{C}_r\, \xi(t), \qquad \mathcal{S}_{\xi_r}(\omega) = \frac{1}{2\pi}\frac{\hbar\omega}{k_b T_r}\left(N_r(\omega) + 1/2\right)$$

We introduce the Green's function of the circuit:

$$\frac{d}{dt}G(t,t') - \mathcal{A}(t)\mathcal{H}(t)G(t,t') = \mathbb{1}\delta(t,t')$$

And we have the formal solution:

$$y(t) = G(t,0)\, y(0) + \int_0^t d\tau\, G(t,\tau) \sum_r \sqrt{2k_b T_r}\, \mathcal{C}_r(\tau)\, \xi(\tau)$$

Using this, we can find that:

$$\begin{aligned}
\sigma(t) &= \langle y(t) y(t)^T \rangle \\
&= G(t,0)\sigma(0)G(t,0)^T + \\
&\int_0^t d\tau \sum_r \sqrt{2k_b T_r}\, \left[G(t,0)\langle y(0)\xi^T(\tau)\rangle \mathcal{C}_r(\tau)^T G(t,\tau)^T + G(t,\tau)\mathcal{C}_r(\tau)\langle \xi(\tau) y(0)^T\rangle G(t,0)^T\right] + \\
&\int_0^t d\tau \int_0^t d\tau' \sum_{r,r'} 2k_b \sqrt{T_r T_{r'}}\, G(t,\tau)\mathcal{C}_r(\tau)\langle \xi(\tau)\xi(\tau')^T\rangle \mathcal{C}_{r'}(\tau')^T G(t,\tau')^T.
\end{aligned}$$

Generalization to quantum noise

Green's function techniques

Differential equation for the covariance matrix:

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r \left(\mathcal{I}_r(t)\, \mathcal{C}_r\mathcal{C}_r^T + \mathcal{C}_r\mathcal{C}_r^T\, \mathcal{I}_r(t)^T\right)$$

where:

$$\mathcal{I}_r(t) = \int_0^t d\tau\, G(t, t-\tau)\, \langle \xi_r(0)\xi_r(\tau)\rangle$$

Classical limit:
In the limit of large temperatures we have $\langle \xi_r(0)\xi_r(\tau)\rangle \to \delta(t-t')$ and therefore

$$\mathcal{I}_r(t) = \int_0^t d\tau\, G(t, t-\tau)\, \langle \xi_r(0)\xi_r(\tau)\rangle \to \frac{1}{2}G(t,t) = \frac{\mathbb{1}}{2}$$

and we recover the classical result:

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r\, \mathcal{C}_r\mathcal{C}_r^T$$

Generalization to quantum noise

Quantum heat currents

Differential equation for the covariance matrix:

$$\frac{d}{dt}\sigma(t) = \mathcal{A}\mathcal{H}(t)\sigma(t) + \sigma(t)\mathcal{H}(t)\mathcal{A}^T + \sum_r 2k_b T_r \left(\mathcal{I}_r(t)\,\mathcal{C}_r\mathcal{C}_r^T + \mathcal{C}_r\mathcal{C}_r^T\,\mathcal{I}_r(t)^T\right)$$

Quantum heat currents:

$$\langle\dot{Q}\rangle = \frac{1}{2}\,\mathrm{Tr}\left[\mathcal{H}(t)\frac{d}{dt}\left(\langle x\rangle\langle x\rangle^T + \sigma\right)\right]$$

Again, under the condition $Q_{\mathsf{RR}} = 0$, we have:

$$\langle\dot{Q}_r\rangle = \langle j_r\rangle\langle v_r\rangle + \mathrm{Tr}\left[(\mathcal{H}\sigma(t)\mathcal{H} - 2k_b T_r\mathcal{H}\mathcal{I}_r(t))\mathcal{C}_r\mathcal{C}_r^T\right]$$

Quantum heat currents in frequency domain

Partial transform of the Green's function

$$\hat{G}(t,\omega) = \int_0^t d\tau\, e^{-i\omega(t-\tau)}\, G(t,\tau) \qquad \Longrightarrow \qquad \frac{d}{dt}\hat{G}(t,\omega) = \mathbb{1} - [i\omega - \mathcal{A}\mathcal{H}(t)]\hat{G}(t,\omega)$$

Important property: $\frac{d}{dt}\left(\hat{G}^\dagger \mathcal{H}\hat{G}\right) - \hat{G}^\dagger \frac{d\mathcal{H}}{dt}\hat{G} - 2\hat{G}^\dagger\mathcal{H}(\mathcal{A})_s\mathcal{H}\hat{G} = \mathcal{H}\hat{G} + \hat{G}^\dagger\mathcal{H}$

Convolution function and covariance matrix

$$\mathcal{I}_r(t) = \frac{1}{2\pi k_b T_r}\int_{-\Lambda}^{+\Lambda} d\omega\, \hbar\omega\, \hat{G}(t,\omega)(N_r(\omega)+1/2)$$

Λ: High frequency cut-off

$$\sigma(t) = \frac{1}{\pi}\sum_r \int_{-\Lambda}^{+\Lambda} d\omega\, \hbar\omega\, \hat{G}(t,\omega)\mathcal{D}_r\hat{G}(t,\omega)^\dagger(N_r(\omega)+1/2)$$

$$\mathcal{D}_r = \mathcal{C}_r\mathcal{C}_r^T$$

We can now enter all this information in our expression for the local heat currents

$$\langle \dot{Q}_r\rangle = \langle j_r\rangle\langle v_r\rangle + \mathrm{Tr}\left[(\mathcal{H}\sigma(t)\mathcal{H} - 2k_b T_r\mathcal{H}\mathcal{I}_r(t))\mathcal{C}_r\mathcal{C}_r^T\right]$$

Quantum heat currents in frequency domain

Generalization of Landauer-Büttiker formula

$$\boxed{\langle \dot{Q}_r \rangle = \langle j_r \rangle \langle v_r \rangle + \sum_{r'} \int_{-\Lambda}^{+\Lambda} d\omega \, \hbar\omega \, f_{r,r'}(t,\omega) \, (N_{r'}(\omega) + 1/2)}$$

Non-diagonal elements: $f_{r,r'}(t,\omega) = \frac{1}{\pi} \operatorname{Tr}\left[\mathcal{H}(t)\hat{G}(t,\omega)\mathcal{D}_{r'}\hat{G}(t,\omega)^\dagger \mathcal{H}(t)\mathcal{D}_r\right] \qquad (r \neq r')$

Sum over first index: $\bar{f}_{r'}(t,\omega) = \sum_r f_{r,r'}(t,\omega) = \frac{1}{2\pi} \operatorname{Tr}\left[\left(\hat{G}^\dagger \frac{d\mathcal{H}}{dt}\hat{G} - \frac{d}{dt}\left(G^\dagger \mathcal{H}\hat{G}\right)\right)\mathcal{D}_{r'}\right]$

Particular case: for static circuits ($\bar{f}_r' = 0$) we recover the usual Landauer-Büttiker formula

$$\langle \dot{Q}_r \rangle = \langle j_r \rangle \langle v_r \rangle + \sum_{r'} \int_{-\Lambda}^{+\Lambda} d\omega \, \hbar\omega \, f_{r,r'}(\omega) \, (N_{r'}(\omega) - N_r(\omega))$$

Main result

We have derived a generalized Landauer-Büttiker formula which is valid for arbitrary circuits, with any number of resistors at arbitrary temperatures, and for arbitrary driving protocols.

Landauer-Büttiker formula for periodic driving

Again, we consider periodic parametric driving: $\mathcal{H}(t) = \sum_{k=-\infty}^{+\infty} \mathcal{H}_k \, e^{ik\omega_d t}$

Then $G(t, \omega)$ is asymptotically periodic: $\hat{G}(t, \omega) = \sum_{j=-\infty}^{+\infty} \hat{G}_j(\omega) \, e^{ij\omega_d t}$, where:

$$i(\omega + j\omega_d)\hat{G}_j(\omega) = \mathbb{1}\delta_{j,0} + \mathcal{A} \sum_k \mathcal{H}_k \hat{G}_{j-k}(\omega)$$

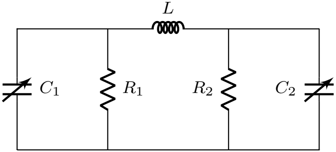
Then:

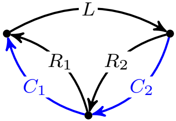
$$\boxed{\langle \dot{Q}_r \rangle_c = \langle \langle j_r \rangle \langle v_r \rangle \rangle_c + \sum_{r'} \int_{-\Lambda}^{+\Lambda} d\omega \, \hbar\omega \, F_{r,r'}(\omega) \, (N_{r'}(\omega) + 1/2)}$$

$$F_{r,r'}(\omega) = \frac{1}{\pi} \sum_{j,j',k} \mathrm{Tr} \left[\mathcal{H}_k \hat{G}_j(\omega) \mathcal{D}_{r'} \hat{G}^\dagger_{j'}(\omega) \mathcal{H}_{j'-j-k} \mathcal{D}_r \right] \text{ for } r' \neq r \qquad (1)$$

$$\bar{F}_{r'}(\omega) = \sum_r F_{r,r'}(\omega) = \frac{1}{2\pi} \sum_{j,k} ik\omega_d \, \mathrm{Tr} \left[\hat{G}^\dagger_j(\omega) \mathcal{H}_k \hat{G}_{j-k}(\omega) \mathcal{D}_{r'} \right] \qquad (2)$$

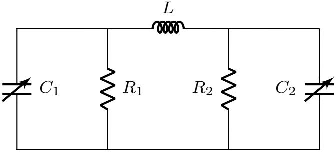
Quantum limits for cooling

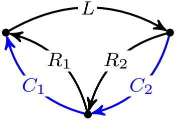




Going back to our cooling scheme, what happens if we enter the quantum regime?

Quantum limits for cooling





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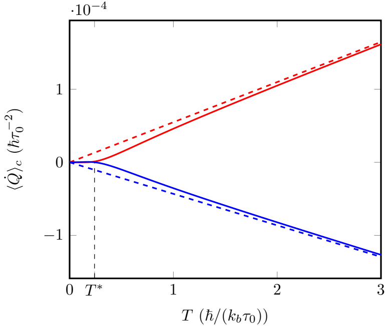
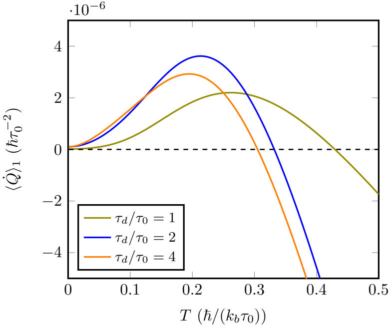




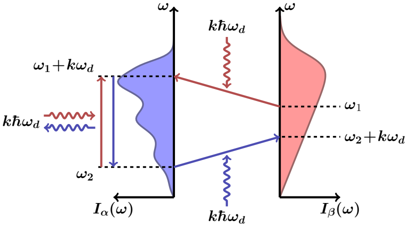
$\tau_0 = \sqrt{LC}$ and $\tau_d = RC$

Quantum limits for cooling

Physical intuition

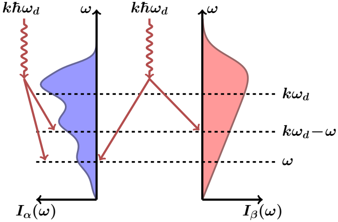
There are two different kind of contributions to the heat currents:

Transport of excitations



- $\langle\dot{Q}\rangle \to 0$ for $T \to 0$
- $\langle\dot{Q}\rangle \propto \gamma$

Pair creation of excitations



- $\langle\dot{Q}\rangle \to 0$ for $T \to 0$
- $\langle\dot{Q}\rangle \propto \gamma^2$

Master equations fail to describe this contribution!

- Freitas, N., Paz JP. Physical Review E 95.1 (2017): 012146.
- Freitas, N., Gallego, R., Masanes, L., Paz, JP. Cooling to Absolute Zero: The Unattainability Principle

Summary and next steps

Stochastic and Quantum Thermodynamics of Driven RLC Networks

Nahuel Freitas,[1] Jean-Charles Delvenne,[1] and Massimiliano Esposito[1]
[1] *Complex Systems and Statistical Mechanics, Physics and Materials Science, University of Luxembourg, L-1511 Luxembourg, Luxembourg*

arXiv:1906.11233

- We have considered the stochastic description of general RLC circuits

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  - It does not require the quantization of the circuit.

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  - It does not require the quantization of the circuit.
- We will employ this formalism to study optimal thermal cycles in large and complex electrical circuits.