Shortcut to Equilibration enabled by the Inertial Theorem

Quantum Thermodynamics – Espoo 2019

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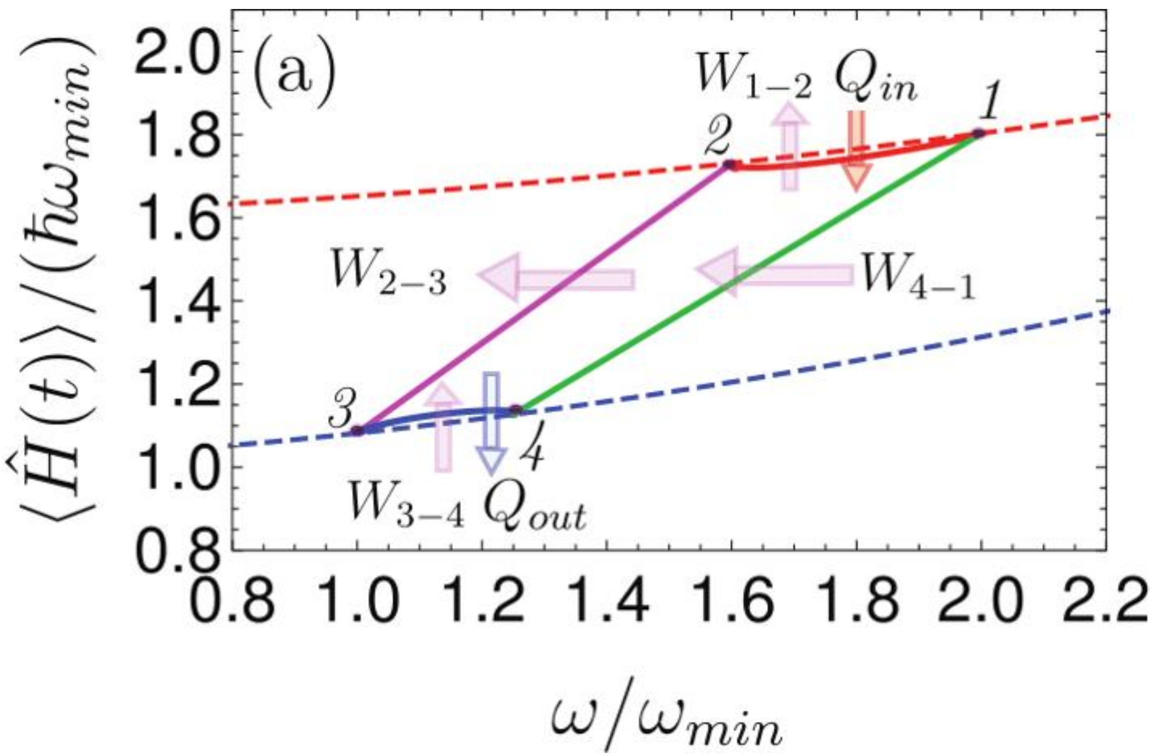
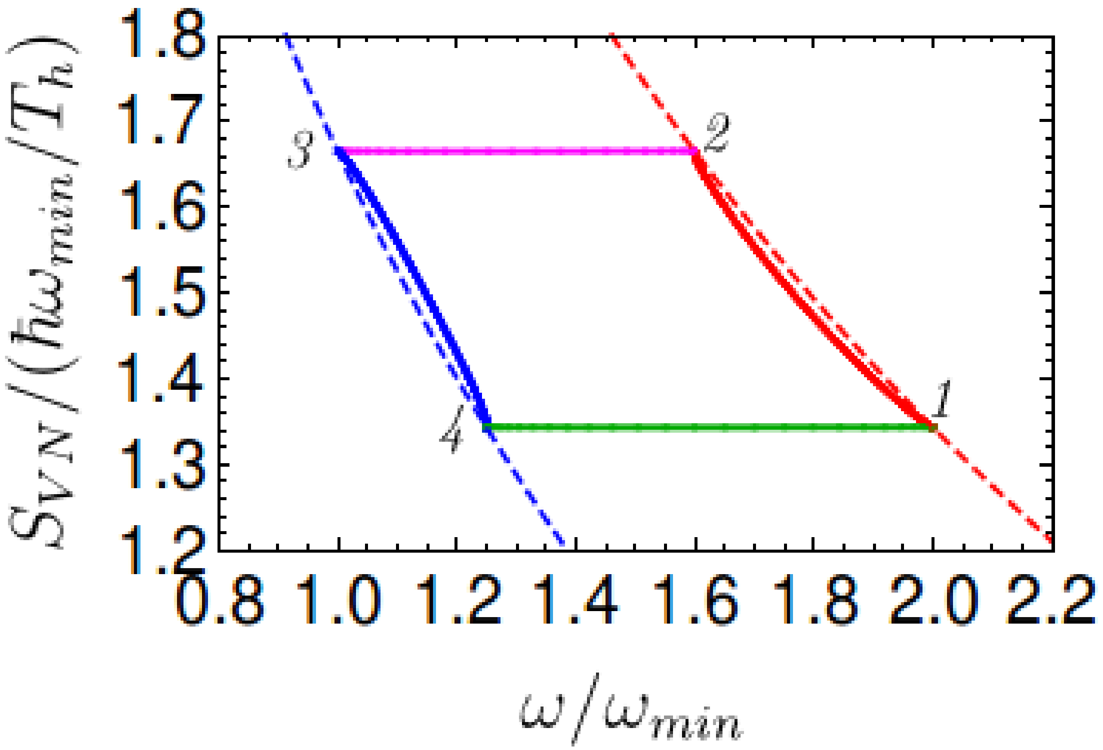
Motivation – Quantum Carnot cycle

Work and Heat simultaneously



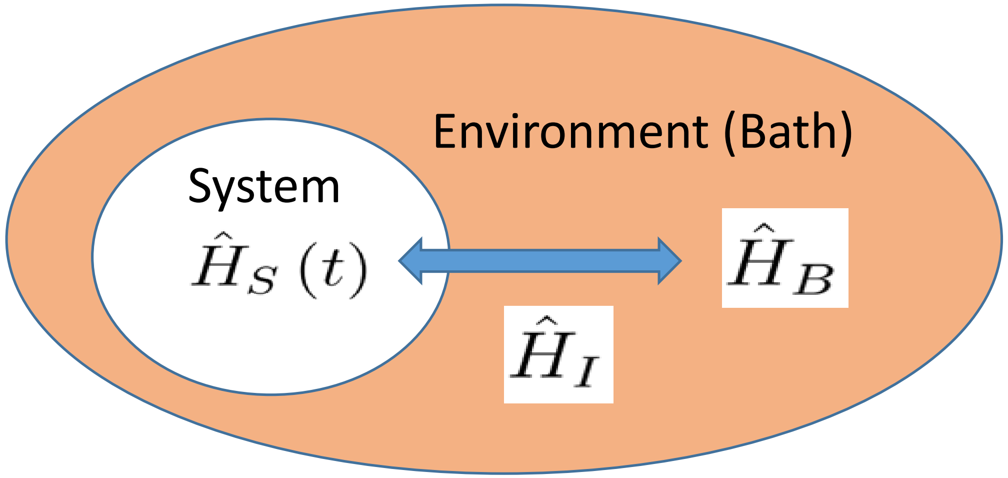


Open dynamics including **non-adiabatic driving**

Quantum Signatures in the Quantum Carnot Cycle, R. Dann and R. Kosloff, *arXiv*:1906.06946

Motivation – Quantum Carnot cycle

$$\hat{H}(t) = \hat{H}_S(t) + \hat{H}_B + \hat{H}_I$$



Open dynamics with **non-adiabatic driving**

Complete description including both **coherence** and **energy**

Quantum Signatures in the Quantum Carnot Cycle, R. Dann and R. Kosloff, *arXiv*:1906.06946

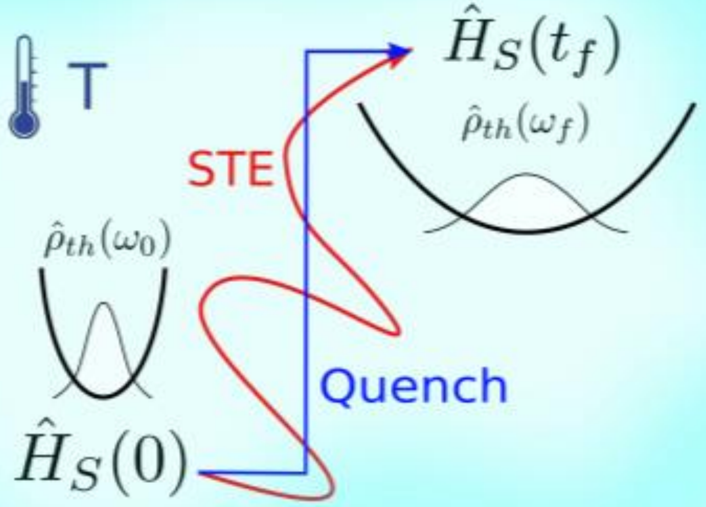
Motivation II – Increase the **power** output of a Carnot-analog cycle

➡ Shortcut
to
equilibration

$$\hat{H}(t) = \hat{H}_S(t) + \hat{H}_B + \hat{H}_I$$

Open system control problem:

How to control a state to state transformation in an open system?



Entropy change

Shortcut to Equilibration of an Open Quantum System, R. Dann, A. Tobalina, and R. Kosloff, *PRL* (2019)

How can we obtain the Master equation?

Analytic tools:

$$\hat{H}(t) = \hat{H}_S(t) + \hat{H}_B + \hat{H}_I$$

Non-Adiabatic Master Equation (NAME)

Inertial theorem

Non-adiabatic open system dynamics

Time-dependent Markovian Master Eq., R. Dann, A. Levy, and R. Kosloff, *Phys. Rev. A* 98, 052129 (2018).
The Inertial Theorem, R. Dann and R. Kosloff, *arXiv*:1810.12094 (2018).

NAME – Time dependent non-adiabatic process

Separation of timescales between system and bath allows reduced system dynamics

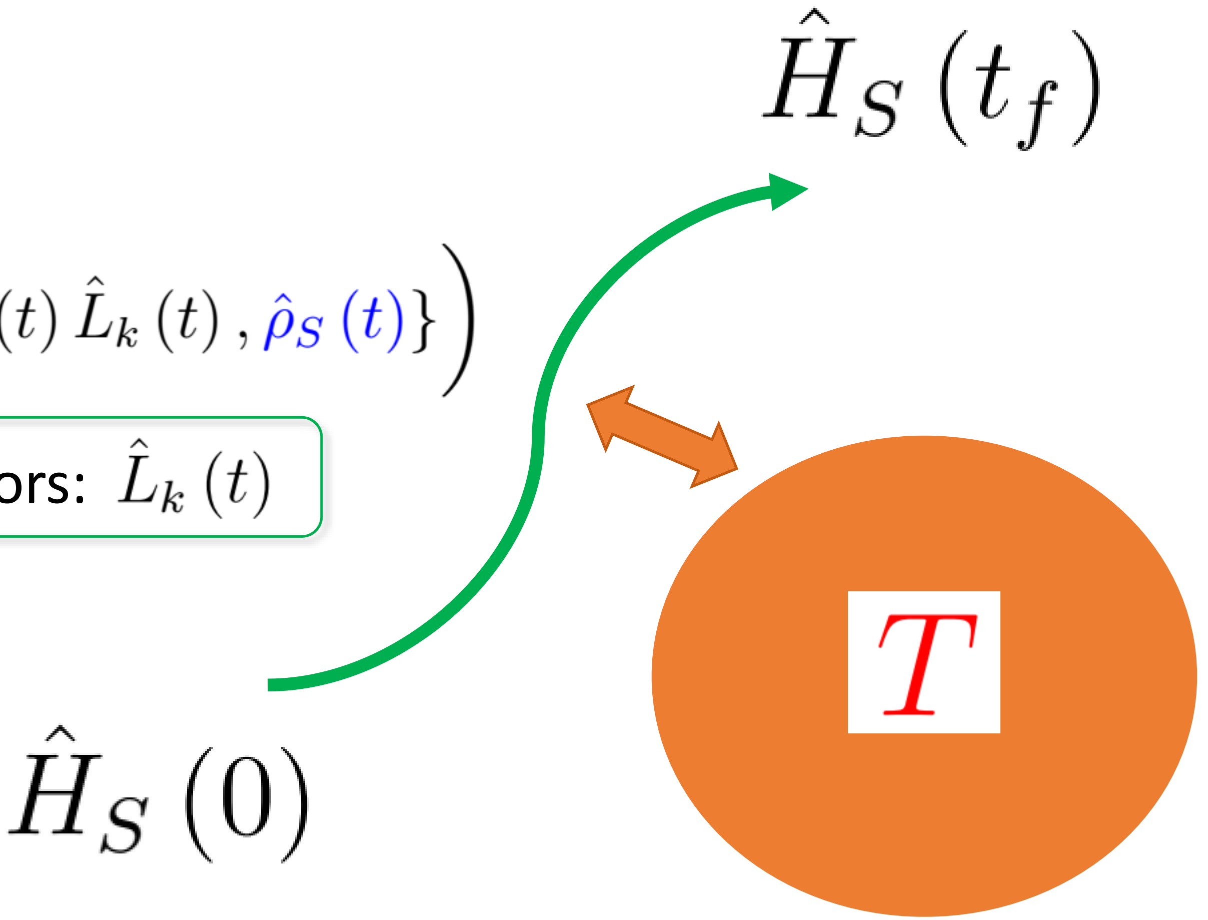
$$\frac{d}{dt}\hat{\rho}_S = \mathcal{L}_S(t)\hat{\rho}_S$$

$$\frac{d}{dt}\hat{\rho}_S(t) = \sum_k c_k(t)\left(\hat{L}_k(t)\,\hat{\rho}_S(t)\hat{L}_k^{\dagger}(t) + \frac{1}{2}\{\hat{L}_k^{\dagger}(t)\,\hat{L}_k(t)\,,\hat{\rho}_S(t)\}\right)$$

Lindblad Jump Operators: $\hat{L}_k(t)$

Where $\mathcal{L}_S$ depends on the bath implicitly.

Not necessarily periodic!



Non Adiabatic Master Equation (NAME)

$$\hat{H}(t) = \hat{H}_S(t) + \hat{H}_B + \hat{H}_I$$

$$\hat{H}_I = \sum_k g_k \hat{A}_k \otimes \hat{B}_k$$

- Following Davies's derivation, the first step is a transformation to the Interaction picture:

$$\tilde{A}_k(t) = \hat{U}_S^\dagger(t)\,\hat{A}_k \hat{U}_S(t) \qquad \tilde{B}_k(t) = e^{i\hat{H}_B t}\hat{B}_k e^{-i\hat{H}_B t}$$

Where the system evolution operator is given by,

$$i\frac{\partial \hat{U}_S(t)}{\partial t} = \hat{H}_S(t)\hat{U}_S(t) \qquad \hat{U}_S(0) = I \qquad \hbar = 1$$

R. Dann, A. Levy, and R. Kosloff, *Phys. Rev. A* 98, 052129 (2018).

How do we find the jump operators ?

Liouville space representation:

Operator Hilbert space with an inner product: $\left(\hat{X}_i, \hat{X}_j\right) \equiv \mathrm{tr}\left(\hat{X}_i^{\dagger} \hat{X}_j\right)$

Time-dependent Hamiltonian: $\hat{H}_S(t)$

$$\mathcal{U}_S(t) = \mathcal{T} e^{i \int_0^t \left(\left[\hat{H}_S(t'), \cdot\right] + \frac{\partial}{\partial t'}\right) dt'}$$

$$\mathcal{U}_S(t) \hat{F}_j(0) = \lambda(t) \hat{F}_j(0)$$

Stationary Hamiltonian: $\hat{H}_S = \sum_i \epsilon_i |i\rangle \langle i|$

$$\mathcal{U}_S(t) = e^{i\left[\hat{H}_S, \cdot\right] t}$$

$$\mathcal{U}_S(t) |n\rangle \langle m| = e^{i(\epsilon_n - \epsilon_m)t} |n\rangle \langle m|$$

Excitations

Wave-function representation:

$$\tilde{F}_j(t) = \hat{U}^{\dagger}(t) \hat{F}_j(0) \hat{U}_S(t) = \lambda_j(t) \hat{F}_j(0)$$

$\Rightarrow$ $\{\hat{F}_j\}$ **Eigenoperators**

Derivation of the NAME

Transformation to the Interaction picture: $\tilde{\rho}(t) = \hat{U}_B^{\dagger}\hat{U}_S^{\dagger}(t)\,\hat{\rho}(t)\,\hat{U}_S(t)\,\hat{U}_B$

$$\rightarrow \quad \frac{d}{dt}\tilde{\rho}(t) = -i\left[\tilde{H}_I(t), \tilde{\rho}(t)\right]$$

If $\hat{H}_I = \sum_k g_k \hat{A}_k \otimes \hat{B}_k$ then we can expand in $\hat{A}_k$

the **eigneoperator** basis $\{\hat{F}_j\}$ $\rightarrow$ $\tilde{A}_k(t) = \sum_j \xi_j^k(t)\, e^{i\theta_j(t)} \hat{F}_j$

Interaction representation

$$\rightarrow \quad \tilde{H}_I(t) = \sum_j \xi_j^k(t)\, e^{i\theta_j(t)} \hat{F}_j \otimes \tilde{B}_k$$

$\hat{F}_j \equiv \hat{F}_j(0)$

Time independent!

$\hat{F}_j$ **become the jump operators**

Non Adiabatic Master Equation (NAME)

$\tau_S = \left(\frac{1}{\omega_i(t)}\right)$ $\tau_B \sim \frac{1}{\Delta\nu}$ $\tau_R \propto (g^2)^{-1}$ τ_d

1. Weak coupling
2. Born- Markov approximation $\tilde{\rho}(t) = \tilde{\rho}_S(t) \otimes \tilde{\rho}_B$
3. Fast bath dynamics relative to the external driving

1. $\tau_B \ll \tau_R$ 2. $\tau_B \ll \tau_S$ 3. $\tau_B \ll \tau_d$

$$\frac{d}{dt}\tilde{\rho}_S(t) = -i\left[\tilde{H}_{LS}(t), \tilde{\rho}_S(t)\right] + \sum_{k,j} \left(\xi_j^k(t)\right)^2 g_k^2 \gamma_{kk}\left(\alpha_j^k(t)\right)\left(\hat{F}_j \tilde{\rho}_S(t)\hat{F}_j^\dagger - \frac{1}{2}\{\hat{F}_j^\dagger \hat{F}_j, \tilde{\rho}_S(t)\}\right)$$

$\hat{F}_j \equiv \hat{F}_j(0)$

Lamb-shift $\tilde{H}_{LS}(t) = \sum_{k,j} \hbar S_{kk}\left(\alpha_j^k(t)\right)\hat{F}_j^\dagger \hat{F}_j$

R. Dann, A. Levy, and R. Kosloff, *Phys. Rev. A 98*, 052129 (2018).

How to solve the free dynamics with driving ?

Inertial Theorem

Non-adiabatic driving → $\left[\hat{H}_S(t), \hat{H}_S(t')\right] \neq 0$

→ Time-ordering problem → **We want to solve this!**

The inertial theorem approximates the evolution of a quantum system, driven by an external field. The theorem is valid for fast driving provided the acceleration rate is small.

Liouville space representation: Elements $\{\hat{X}\}$ with inner product $\left(\hat{X}_i, \hat{X}_j\right) \equiv \mathrm{tr}\left(\hat{X}_i^\dagger \hat{X}_j\right)$

Operator basis: $\vec{v}(t) = \{\hat{X}_1(t), \ldots, \hat{X}_N(t)\}$

$$\frac{d}{dt}\vec{v}(t) = \left(i\left[\hat{H}_S(t), \bullet\right] + \frac{\partial}{\partial t}\right)\vec{v}(t) \xrightarrow{\text{Closed Algebra}} \frac{d}{dt}\vec{v}(t) = -i\mathcal{M}(t)\vec{v}(t)$$

For the appropriate driving protocol and operator basis, we can **factorize the generator**

$$\mathcal{M}(t) = \Omega(t)\,\mathcal{B}(\vec{\chi})$$

$$\vec{\chi} = \{\chi_1, \ldots, \chi_r\}$$

The Inertial Theorem, R. Dann and R. Kosloff, *arXiv*:1810.12094 (2018).

Inertial Theorem: For slowly varying $\vec{\chi}$:

Inertial solution

$$\vec{v}\left(\chi,\theta\right)=\sum_{k}c_{k}e^{-i\int_{\theta_0}^{\theta}d\theta'\lambda_{k}}e^{i\phi_{k}}\vec{F}_{k}\left(\vec{\chi}\left(\theta\right)\right)$$

Geometric phase

$$\phi_{k}\left(\theta\right)=i\int_{\vec{\chi}(\theta_0)}^{\vec{\chi}(\theta)}d\vec{\chi}\left(\vec{G}_{k}|\nabla_{\vec{\chi}}\vec{F}_{k}\right)$$

$$\mathcal{M}\left(t\right)=\Omega\left(t\right)\mathcal{B}\left(\vec{\chi}\right)$$

$$\theta\left(t\right)=\int_{0}^{t}\Omega\left(t'\right)dt'$$

$\vec{G}_k$ - bi-orthoganal partners of $\vec{F}_k$

Inertial condition

$$\Upsilon\ll 1$$

Inertial parameter

$$\Upsilon=\max_{\vec{\chi}}\left[\frac{\left(\vec{G}_{k},\nabla_{\vec{\chi}}\mathcal{B}\vec{F}_{n}\right)}{\left(\lambda_{n}-\lambda_{k}\right)^{2}}\left(\frac{d\vec{\chi}}{d\theta}\right)^{2}\right]\ll 1$$

The Inertial Theorem, R. Dann and R. Kosloff, *arXiv*:1810.12094 (2018).
Experimental Verification of the Inertial Theorem, C.K.Hu, R.Dann, *et al.* , *arXiv*:1903.00404

Explicit example – quantum Harmonic oscillator

$$\hat{H}_S = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2(t)\,\hat{Q}^2$$

Liouville time-dependent operator basis

$$\vec{v}(t) = \{\hat{H}_S, \hat{L}, \hat{C}, \hat{K}, \hat{J}, \hat{I}\}^T$$

$$\hat{L} = \frac{\hat{P}^2}{2m} - \frac{1}{2} m\omega^2(t)\,\hat{Q}^2 \qquad \hat{C} = \frac{\omega(t)}{2}\left(\hat{P}\hat{Q} + \hat{Q}\hat{P}\right)$$

$$\hat{K} = \sqrt{\omega(t)}\,\hat{Q} \qquad \hat{J} = \frac{\hat{P}}{m\sqrt{\omega(t)}}$$

Closed Algebra

Heisenberg equation

$$\frac{d\vec{v}(t)}{dt} = \mathcal{M}(t)\,\vec{v}(t) = -i\omega(t)\,\mathcal{B}\vec{v}(t)$$

Protocol: $\omega(t) = \dfrac{\omega(0)}{1 - \mu\omega(0)\,t}$

$$\vec{\chi} = \mu = \frac{\dot{\omega}}{\omega^2}$$

$$\mathcal{B} = i\begin{bmatrix} \mu & -\mu & 0 & 0 & 0 & 0 \\ -\mu & \mu & -2 & 0 & 0 & 0 \\ 0 & 2 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{2} & 1 & 0 \\ 0 & 0 & 0 & -1 & -\frac{\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Inertial Theorem, R. Dann and R. Kosloff, *arXiv*:1810.12094 (2018).

Inertial solution – quantum Harmonic oscillator

$$\frac{d\vec{v}(t)}{dt} = -i\omega(t)\mathcal{B}(\mu)\vec{v}(t)$$

Inertial solution

$$\vec{v}(t) = \sum_k c_k e^{-i\int_{\theta(0)}^{\theta(t)} d\theta' \lambda_k} e^{i\phi_k} \vec{b}_k(\mu(t))$$

$\hat{U}_S(t)$

Geometric phase

$$\phi_k(\theta(t)) = i\int_{\mu(\theta(0))}^{\mu(\theta(t))} d\mu \left(\vec{g}_k | \nabla_\mu \vec{b}_k\right)$$

Eigenvectors of $\mathcal{B}(\mu)$ = Eigen operators $\hat{b}_k$

Inertial condition

$$\Upsilon \sim \frac{1}{(2\kappa)^2 \omega(t)} \frac{d\mu}{dt} \ll 1$$

$$\kappa = \sqrt{4-\mu^2} \qquad \theta(t) = \int_0^t \omega(t')\, dt'$$

In comparison, Adiabatic condition for HO: $\mu \ll 1$

The Inertial Theorem, R. Dann and R. Kosloff, *arXiv*:1810.12094 (2018).

Reduced driven system dynamics – incorporating the NAME and the Inertial Theorem

Transforming variables in the interaction picture:

$$\left[\tilde{b},\tilde{b}^{\dagger}\right]=1$$

$$\frac{d}{dt}\tilde{\rho}_S(t)=k_{\downarrow}(t)\left(\hat{b}\tilde{\rho}_S(t)\hat{b}^{\dagger}-\frac{1}{2}\{\hat{b}^{\dagger}\hat{b},\tilde{\rho}_S(t)\}\right)+k_{\uparrow}(t)\left(\hat{b}^{\dagger}\tilde{\rho}_S(t)\hat{b}-\frac{1}{2}\{\hat{b}\hat{b}^{\dagger},\tilde{\rho}_S(t)\}\right)$$

$$k_{\downarrow}(t)=k_{\uparrow}(t)\,e^{\alpha(t)/k_BT}=\frac{\alpha(t)\,|\vec{d}|^2}{8\pi\varepsilon_0\hbar c}\left(1+N(\alpha(t))\right)$$

$$\alpha(t)=\sqrt{1-\frac{1}{4}\left(\frac{\dot{\omega}(t)}{\omega^2(t)}\right)^2}\,\omega(t)$$

$$\hat{b}\equiv\hat{b}(0)=\sqrt{\frac{m\omega(0)}{2\hbar}}\frac{(\kappa+i\mu)}{\kappa}\left(\hat{Q}(0)+\frac{\mu+i\kappa}{2m\omega(0)}\hat{P}(0)\right)$$

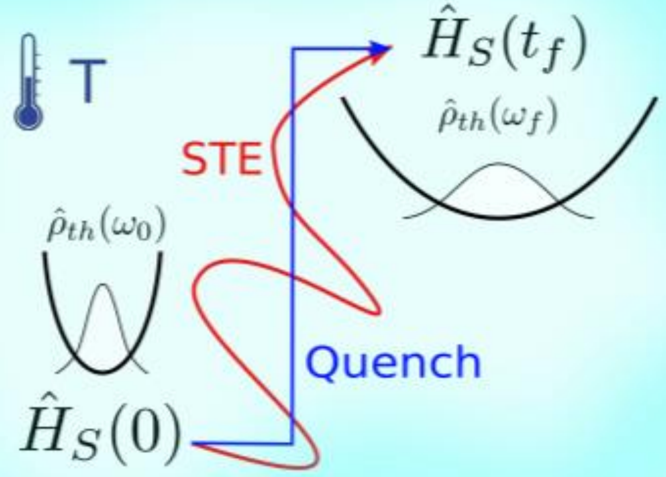
$$\kappa=\sqrt{4-\mu^2}$$

Andersen, H. C., Oppenheim, I., Shuler, K. E., & Weiss, G. H., *Jour. of Math. Phys.* (1964)

Shortcut to Equilibration

$$\hat{H}(t) = \hat{H}_S(t) + \hat{H}_B + \hat{H}_I$$

$$\hat{\rho}_S^{th}(\omega_0) \longrightarrow \hat{\rho}_S^{th}(\omega_f)$$



Entropy change

Shortcut to Equilibration of an Open Quantum System, R. Dann, A. Tobalina, and R. Kosloff, *PRL* (2019)

Engineering the **shortcut to equilibration** protocol

Guessing a **solution** of the Generalized Canonical form

$$\tilde{\rho}_S(t) = (Z(t))^{-1} e^{\gamma(t)\tilde{b}^2} e^{\beta(t)\tilde{b}^\dagger\tilde{b}} e^{\gamma^*(t)\left(\tilde{b}^\dagger\right)^2}$$

$$\dot{\beta} = k_\downarrow\left(e^\beta - 1\right) + k_\uparrow\left(e^{-\beta} - 1 + 4e^\beta|\gamma|^2\right)$$

$$\dot{\gamma} = (k_\downarrow + k_\uparrow)\gamma - 2k_\uparrow\gamma e^{-\beta}$$

$$\dot{\beta} = k_\downarrow(\alpha(t))\left(e^\beta - 1\right) + k_\uparrow(\alpha(t))\left(e^{-\beta} - 1\right)$$

For an initial thermal state

$$\beta(0) = -\frac{\hbar\omega(0)}{k_B T} \qquad \beta(t_f) = -\frac{\hbar\omega(t_f)}{k_B T}$$

$$\mu(0) = \mu(t_f) = 0$$

$$y = e^\beta$$

$$y(s) = y(0) + c_3 s^3 + c_4 s^4 + c_5 s^5$$

$$s = t/t_f$$

$$\alpha(t) = \sqrt{1 - \frac{1}{4}\left(\frac{\dot{\omega}(t)}{\omega^2(t)}\right)^2}\,\omega(t) \longleftarrow \beta(t) \longleftarrow$$

$$\frac{d}{dt}\tilde{\rho}_S(t) \rightarrow \dot{\beta} \rightarrow \beta(t) \rightarrow \alpha(t) \rightarrow \omega(t)$$

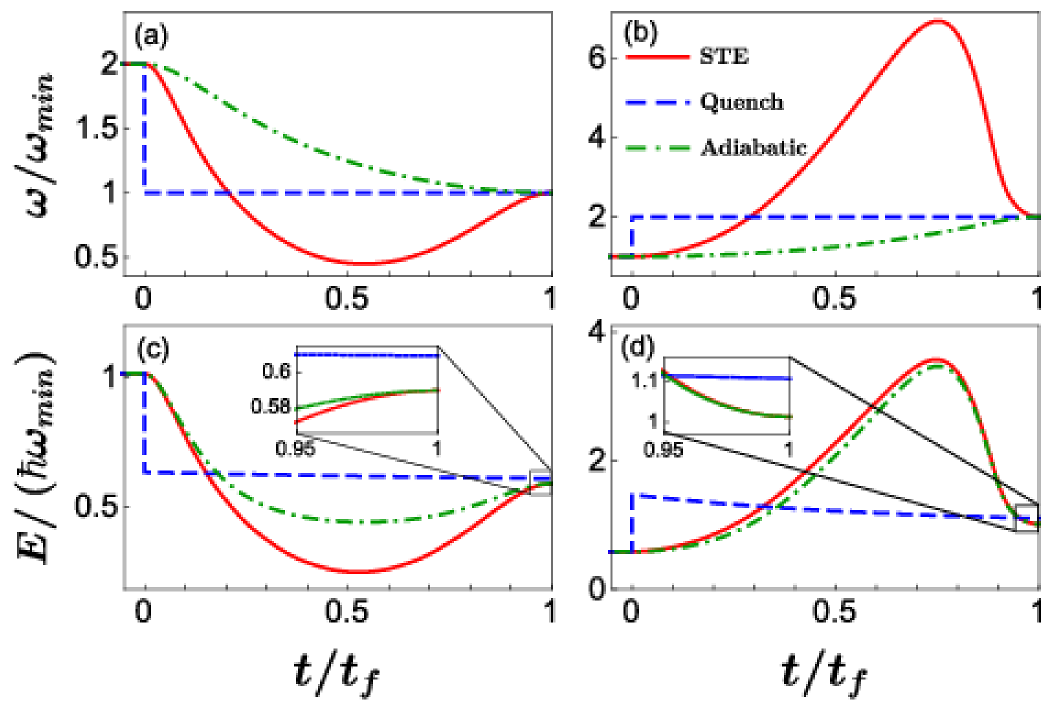
$$\hat{H}_S = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{Q}^2$$

Control

R. Dann, A. Tobalina, and R. Kosloff, arXiv:1812.08821 (2018).

STE - Results

Expansion Compression

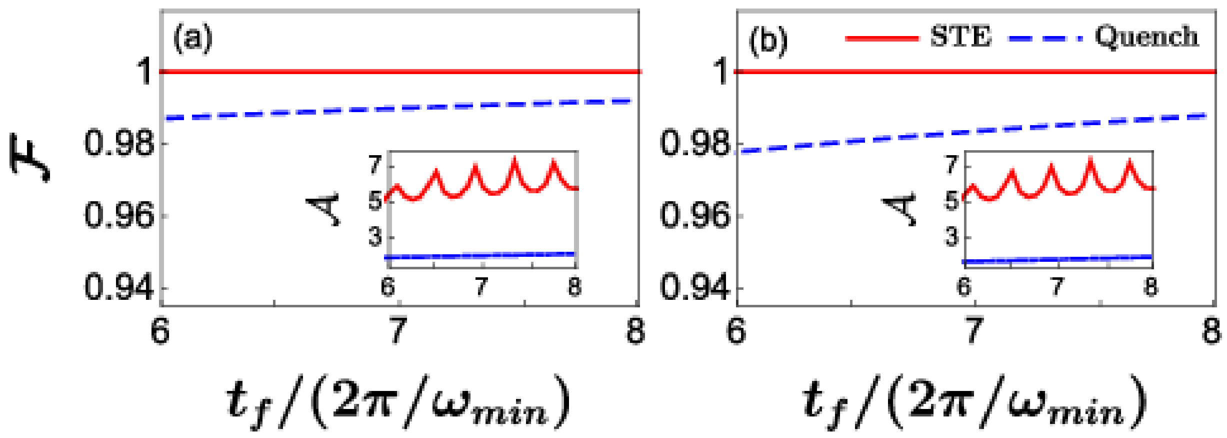


STE Quench Adiabatic

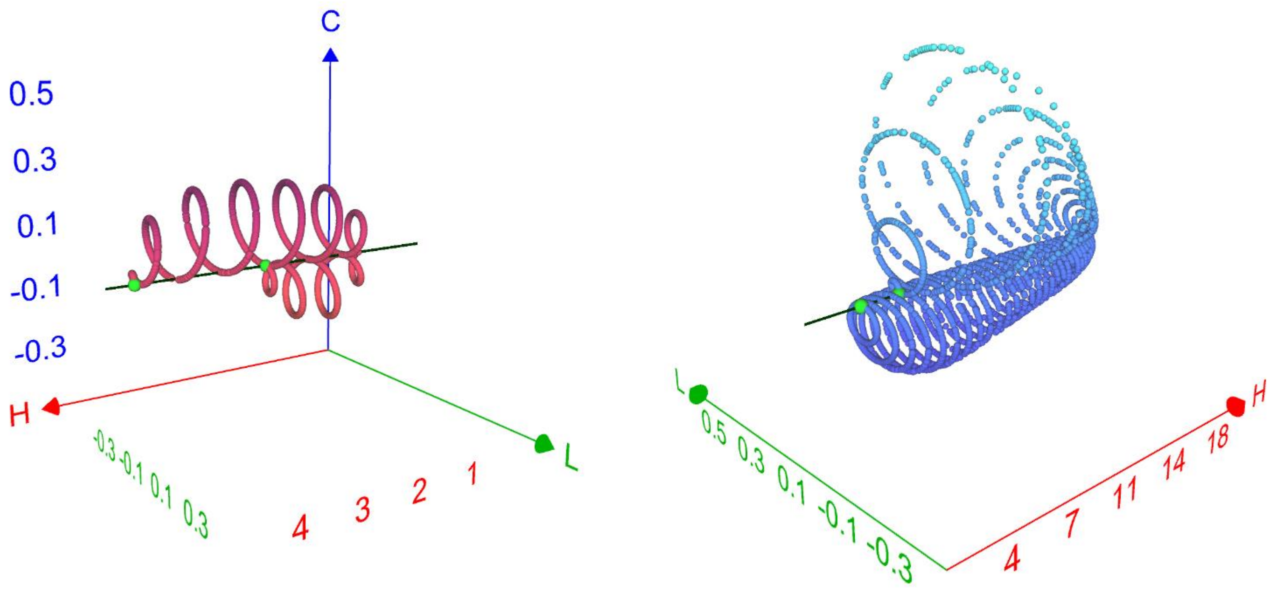
$$Coh \equiv \omega^{-1}\sqrt{\langle\hat{L}\rangle^2 + \langle\hat{C}\rangle^2}$$

Expansion Compression

$$\mathcal{A} = -\log_{10}(1-\mathcal{F})$$



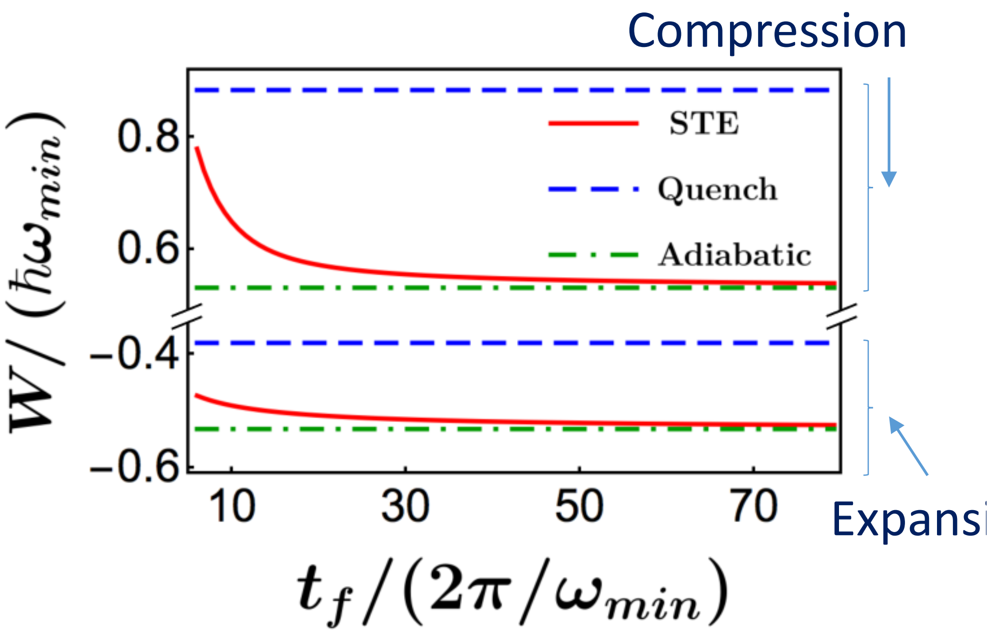
3 fold improvement in time



R. Dann, A. Tobalina, and R. Kosloff, *PRL* (2019)

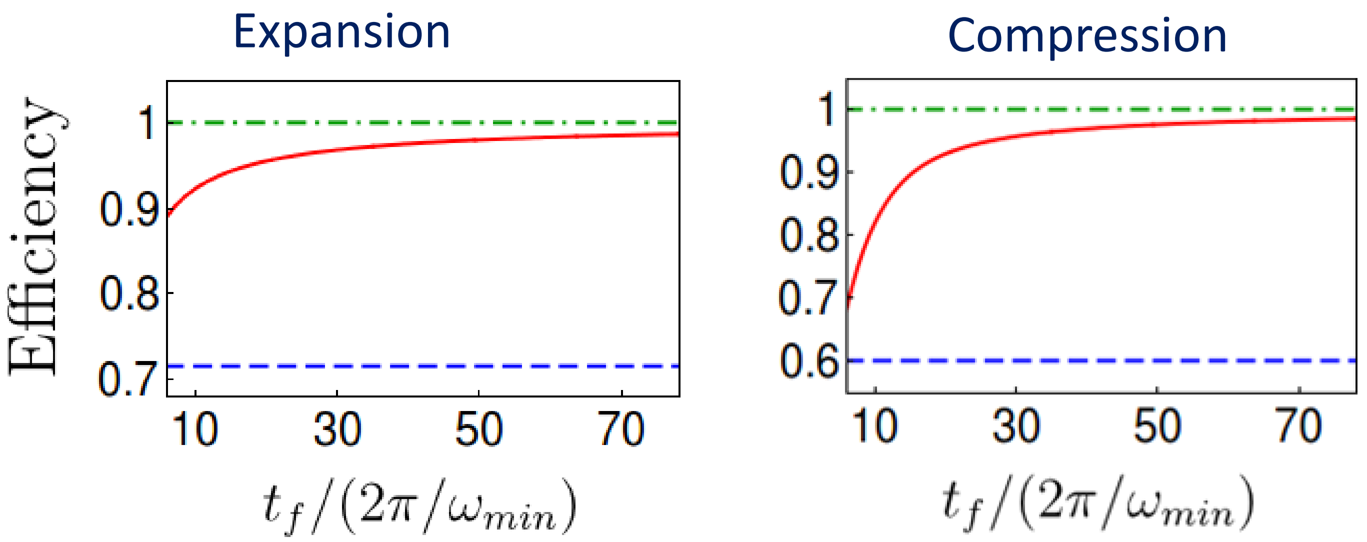
STE- How much does it cost?

STE Quench Adiabatic



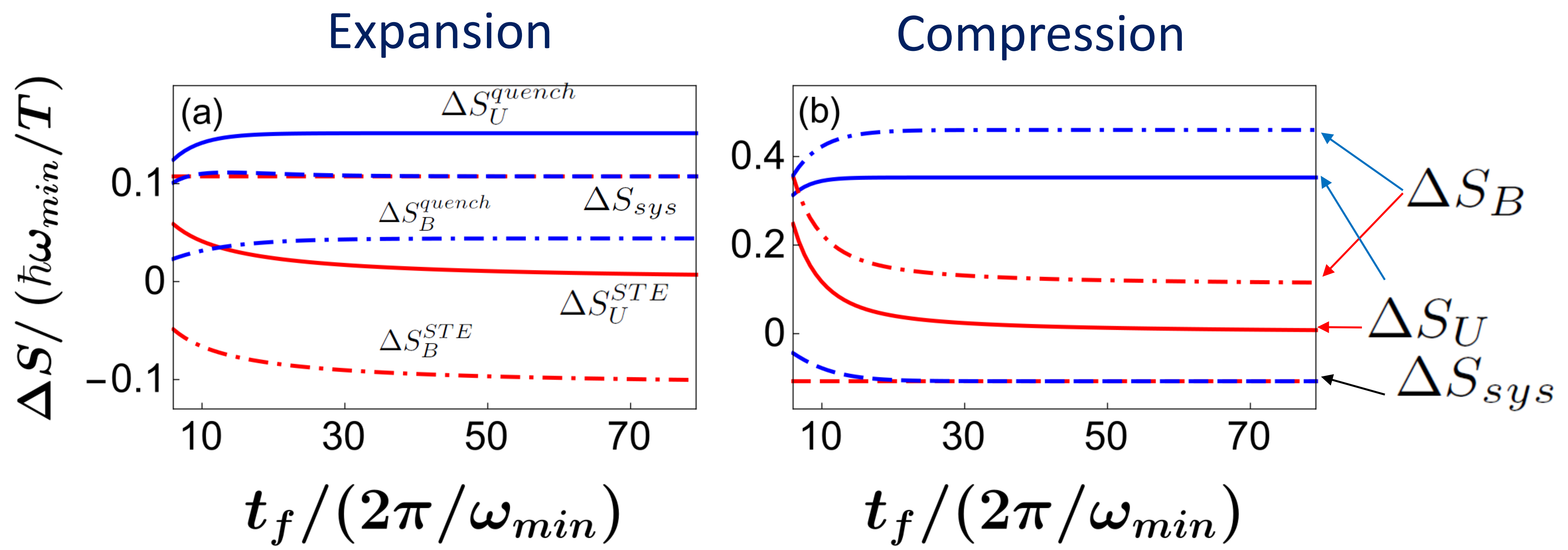
$$W=\int_0^t dt' \, \mathrm{tr}\left(\frac{\partial \hat{H}(t')}{\partial t'}\hat{\rho}_S(t')\right)$$

Rapid driving costs!



Efficiency: $\dfrac{W}{W_{ideal}}$

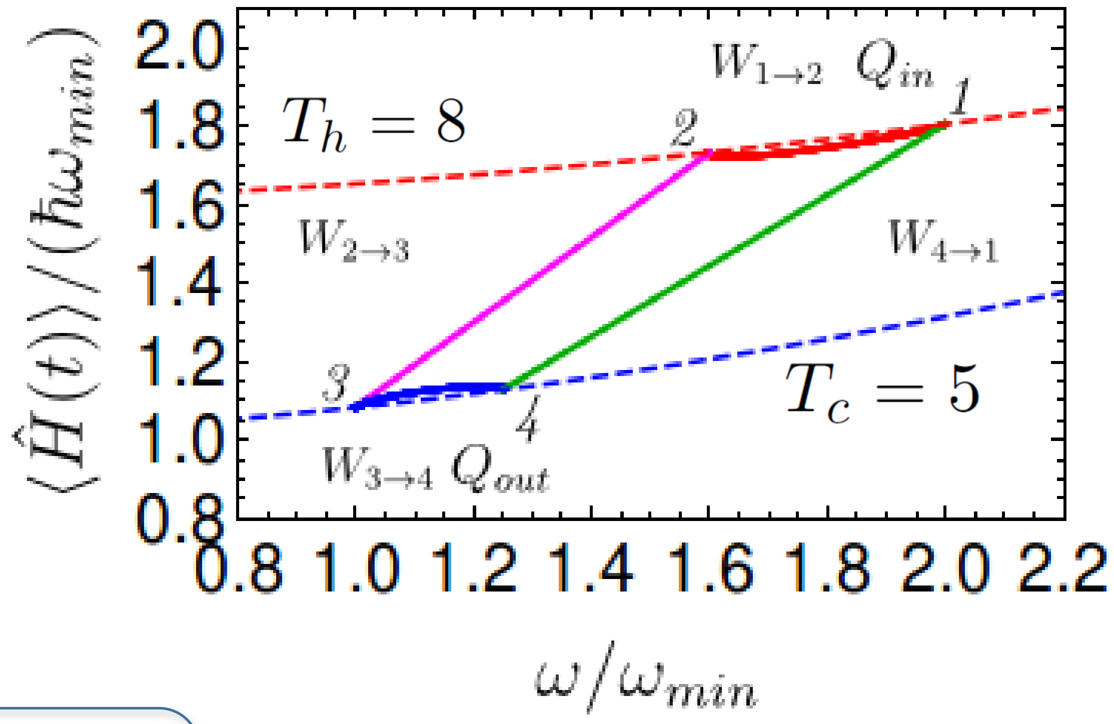
STE – Thermodynamic analysis



STE **Quench**

Quantum Carnot-analog cycle

$$\tau_{cycle} = 250/\left(2\pi/\omega_{min}\right)$$

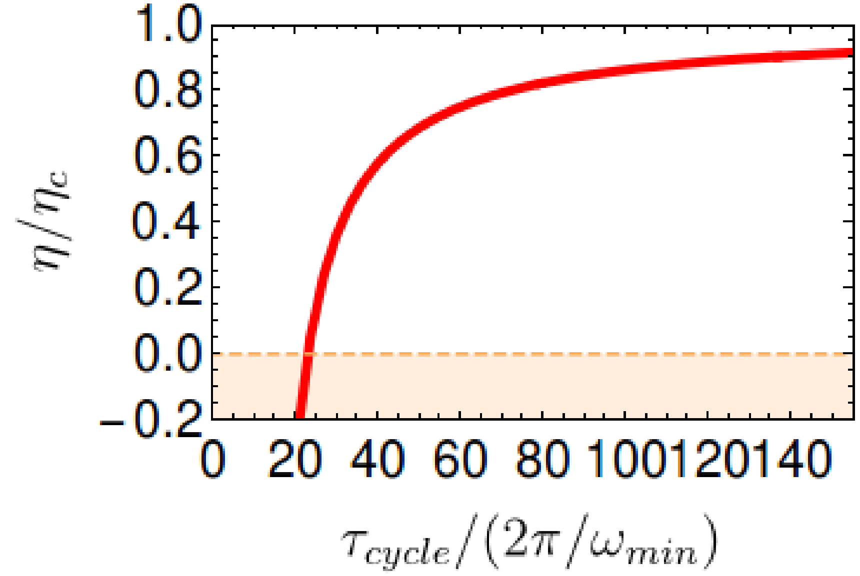


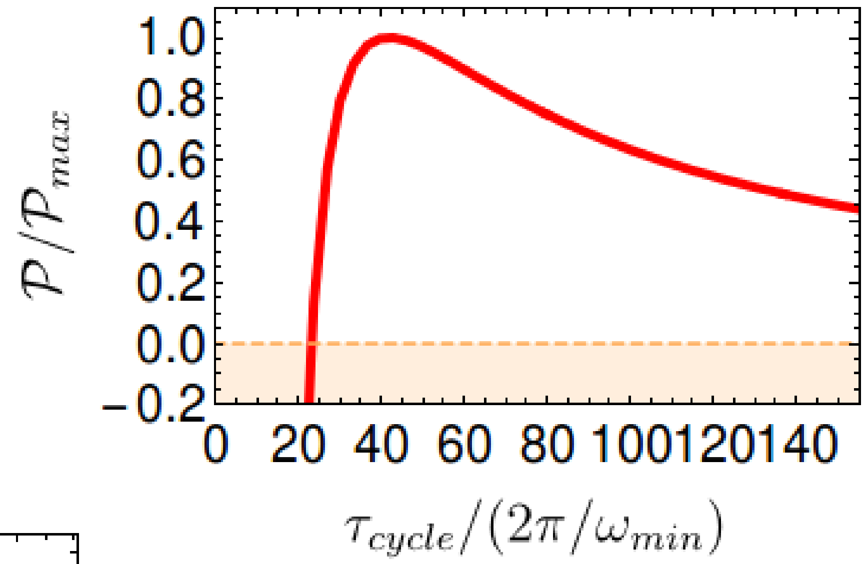
- Open expansion
- Adiabatic expansion
- Open compression
- Adiabatic compression

$$\eta_c = 0.375$$
$$\omega_{min} = 5$$

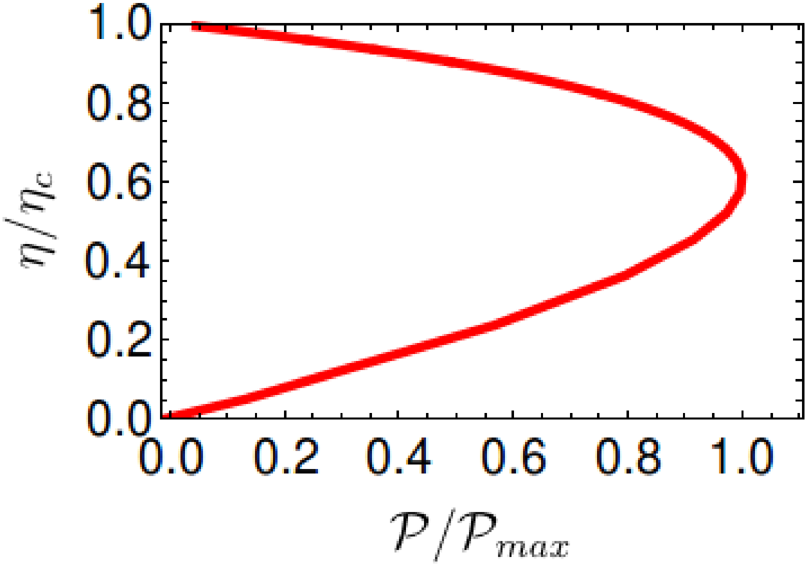
Quantum Signatures in the Quantum Carnot Cycle, R. Dann and R. Kosloff, *arXiv*:1906.06946

Quantum Carnot-analog cycle





$$\eta_{CA} = 0.56\eta_C$$



Efficiency vs Power tradeoff

Quantum Signatures in the Quantum Carnot Cycle, R. Dann and R. Kosloff, *arXiv*:1906.06946

The Journey we took:

Desire: Q. Carnot → Open system dynamic: NAME → Closed system dynamic: Inertial theorem

→ Control problem: STE → Q. Carnot-analog cycle

Shortcut to Equilibration - Study implications

- Quantum Carnot-Analog cycle
- Open quantum system control – rapid change of the system entropy
- Cooling