# Heat bath algorithmic cooling with thermal operations

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# How does one cool/purify qubits?

$$(1-\epsilon)|0\rangle\langle 0| + \epsilon\Pi_{\perp}$$

Depending on implementation, simple ideas, e.g. projective measurements on the system, may not yield purities above the required ones, and other ideas are needed.

# Overview of HBAC

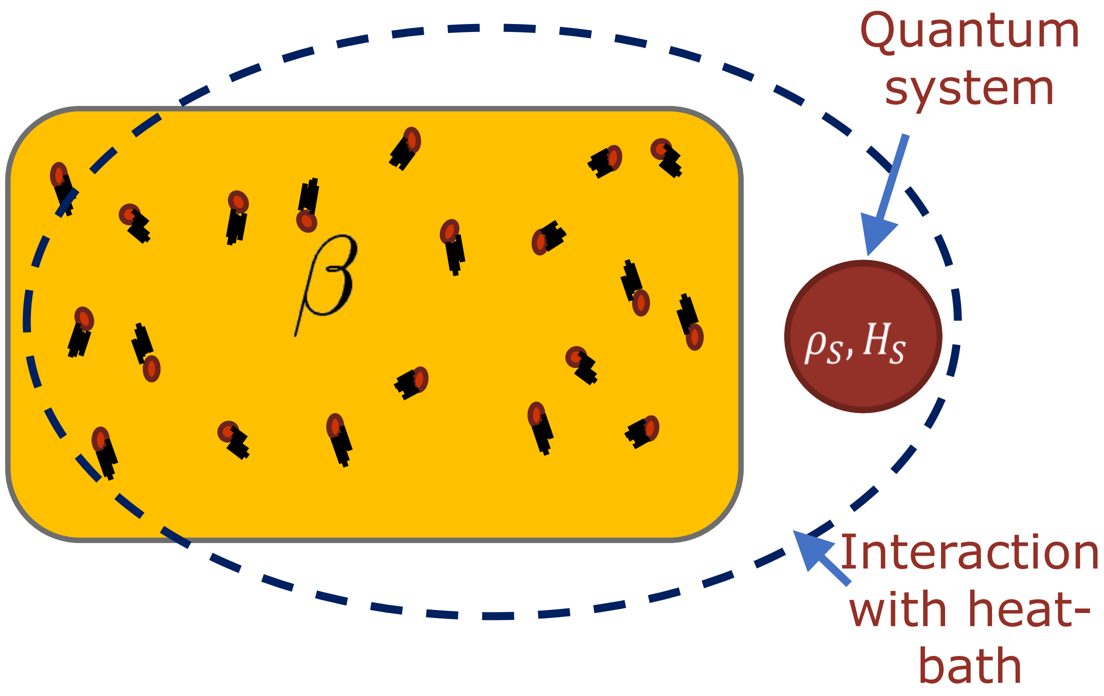
| Scheme | Allowed operations on S and A | A provably optimal algorithm? |
|---|---|---|
| Algorithmic cooling (Schulman & Vazirani ’99) | -Unitaries on S+A | Compression unitary |
| Heath bath algorithmic cooling (Boykin et al ‘02) | -Unitaries on S+A<br>-Thermal reset of A | PPA algorithm:<br>-Numerical : Schulman-Mor-Weinstein ‘05<br>-Analytical: Rodriguez-Briones & Laflamme ‘16 |
| SR- $\Gamma_2$ (Rodriguez-Briones et al ‘17) | -Unitaries on S+A<br>-Thermal reset of virtual qubit (on S+A, but fewer A needed) | ----- |
| HBAC with TO (this talk) | -Unitaries<br>-Thermal operations (generalized thermal step) | Pauli X + $\beta$-swap |

Purity 1 with infinite ancillas

Purity 1 with no ancillas (but infinite rounds)

# HBAC with thermal operations

- Consider (& optimize over) more general interactions with the bath
- Beyond "weak coupling" and Markovian interactions
- Thermal operation: the free operations of the resource theory of athermality
  (Janzig et al. '00, Brandao et al. '13, Horodecki & Oppenheim '13)




These maps are of the form

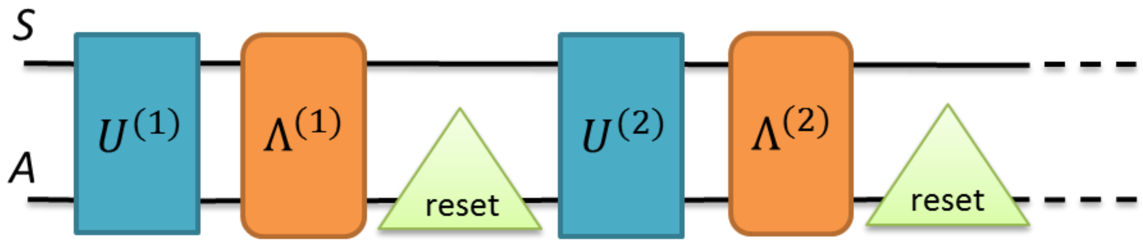
$$\mathcal{E}(\rho_S) = \mathrm{Tr}_B\left[U_{SB}\rho_S \otimes \frac{e^{-\beta H_B}}{Z_B} U_{SB}^\dagger\right]$$

Such that $$[H_S + H_B, U_{SB}] = 0$$

- Natural system-bath set of interactions (thermalizing)
- Convex set
- Fairly well-understood structure
- Some are "simple", some are (really) not

# HBAC with thermal operations

- Protocol round:
  1. Unitary on target + (perhaps) ancillas
  2. Thermal operation on target and/or ancillas
  3. Repeat




- What is the <u>optimal</u> unitary? Is it simple?
- What is the <u>optimal</u> thermal operation? How can it be implemented?
- What  type of interaction Hamiltonian and bath do we need?
- How does it compare to previous schemes?

# Cooling qubits

- With the extra power of TO we do not need ancillas
- The optimal protocol:

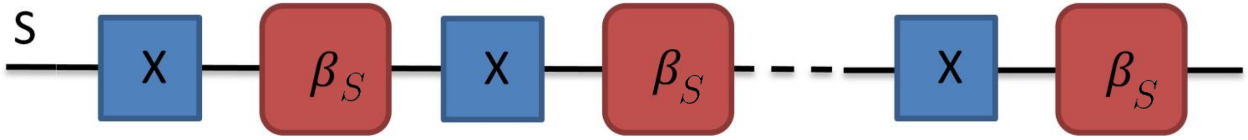
(1) Start in thermal state: $\begin{pmatrix} p_0^{(0)} & 0 \\ 0 & p_1^{(0)} \end{pmatrix} = \frac{1}{1+e^{-\beta\Delta}} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta\Delta} \end{pmatrix}$

(2) Unitary: X (population inversion) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(3) Thermal operation: $\beta$ -swap (stochastic matrix, extremal TO) $\beta_S = \begin{pmatrix} 1-e^{-\beta\Delta} & 1 \\ e^{-\beta\Delta} & 0 \end{pmatrix}$

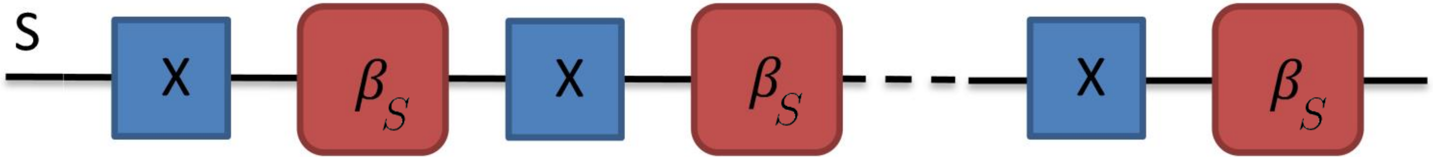
(4) Repeat (2)-(3) $k$ times

$p_0^{(0)}$ ➡




➡ $p_0^{(k)} = 1 - e^{-k\beta\Delta}(1 - p_0^{(0)})$

# Cooling qubits




Iterations of these are optimal (no other sequence cools better)

$$\boxed{p_0^{(k)} = 1 - e^{-k\beta\Delta}(1 - p_0^{(0)})}$$

# Cooling qubits: implementation

- $\beta$ -swap can be implemented with bath a thermal harmonic oscillator $H_B = \Delta a^\dagger a$

(Aberg '14)

$$\beta_S = \begin{pmatrix} 1 - e^{-\beta\Delta} & 1 \\ e^{-\beta\Delta} & 0 \end{pmatrix} \qquad p_0^{(k)} \xrightarrow{k\to\infty} 1$$

- A possible interaction Hamiltonian that generates it

$$H_{JC} = g\big(\sigma_+ \otimes (aa^\dagger)^{-1/2} a + \sigma_- \otimes (aa^\dagger)^{-1/2} a^\dagger\big)$$

- Imperfections on TO $\beta$-swap mean: $\lambda \in [0,1]$

$$\beta_S(\lambda) = \begin{pmatrix} 1 - \lambda e^{-\beta\Delta} & \lambda \\ \lambda e^{-\beta\Delta} & 1 - \lambda \end{pmatrix} \qquad p_0^{(k)} \xrightarrow{k\to\infty} 1 - \frac{1-\lambda}{2 - \lambda(1 + e^{-\beta\Delta})}$$

- Can be done with standard Jaynes-Cummings!!

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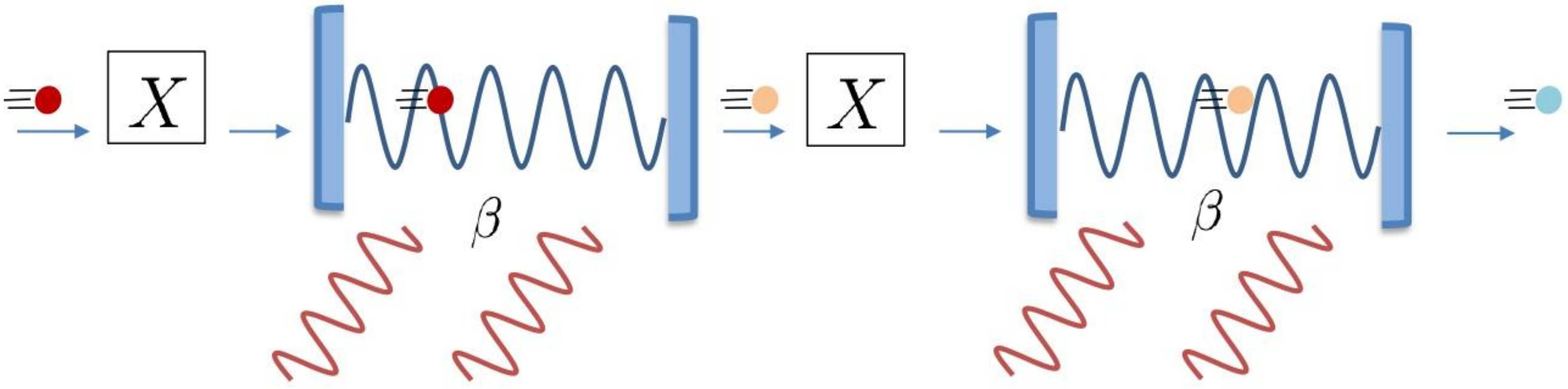
- Can be done with standard Jaynes-Cummings!!

$$H_{JC} = g\big(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger\big)$$

Non-Markovianity is needed to achieve large $\lambda$ and cooling since

$$\lambda_{\text{Markovian}} \leq \frac{1}{1 + e^{-\beta\Delta}} \qquad p_0^{(k)} \xrightarrow{k\to\infty} \frac{1}{1 + e^{-\beta\Delta}}$$

# Cooling qubits: a proposal

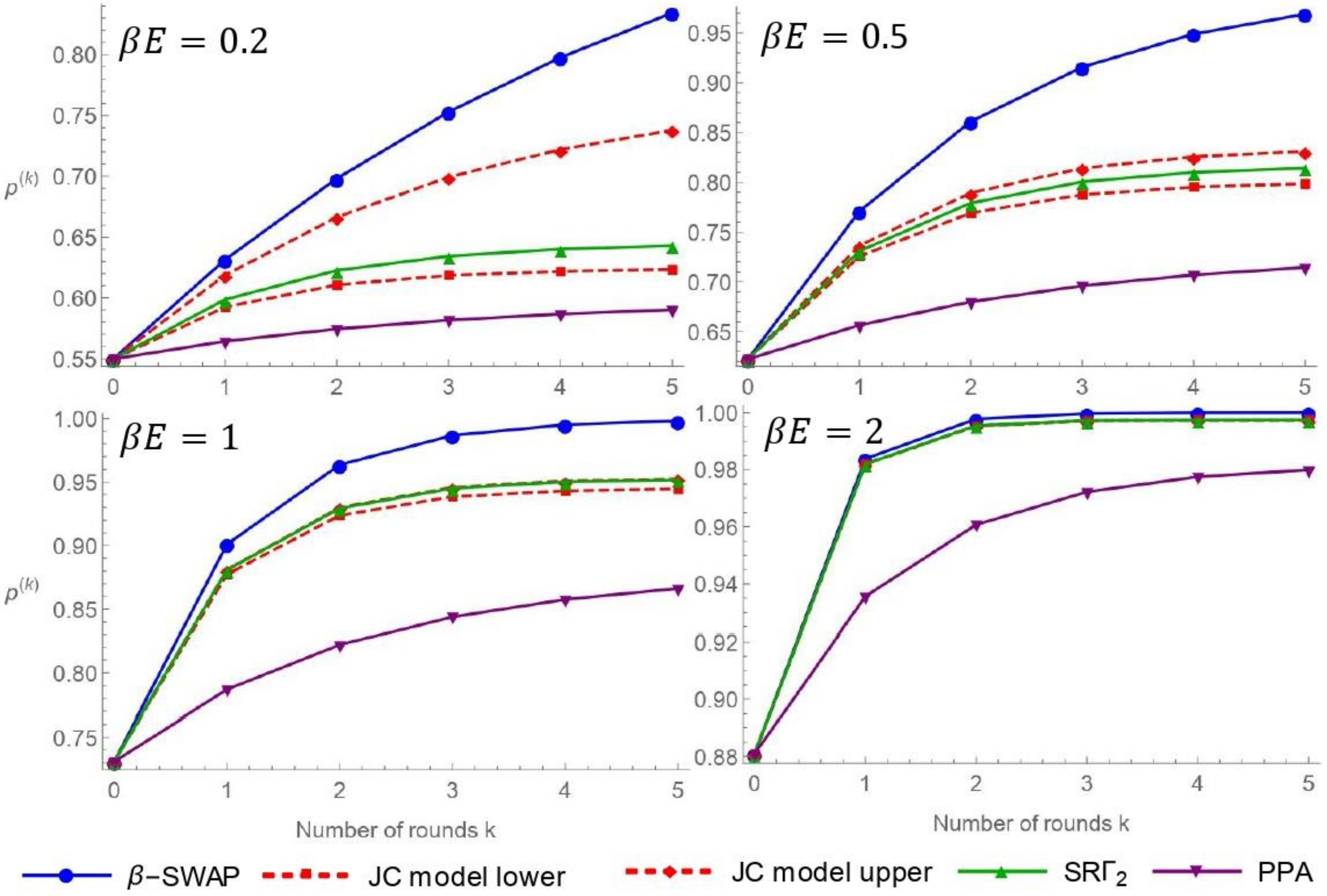



Higher $\lambda$ $\longrightarrow$ Better cooling

(similar requirements as a micromaser, see e.g. Filipowicz et al. PRA ’86)

# Cooling qubits: comparison of HBAC

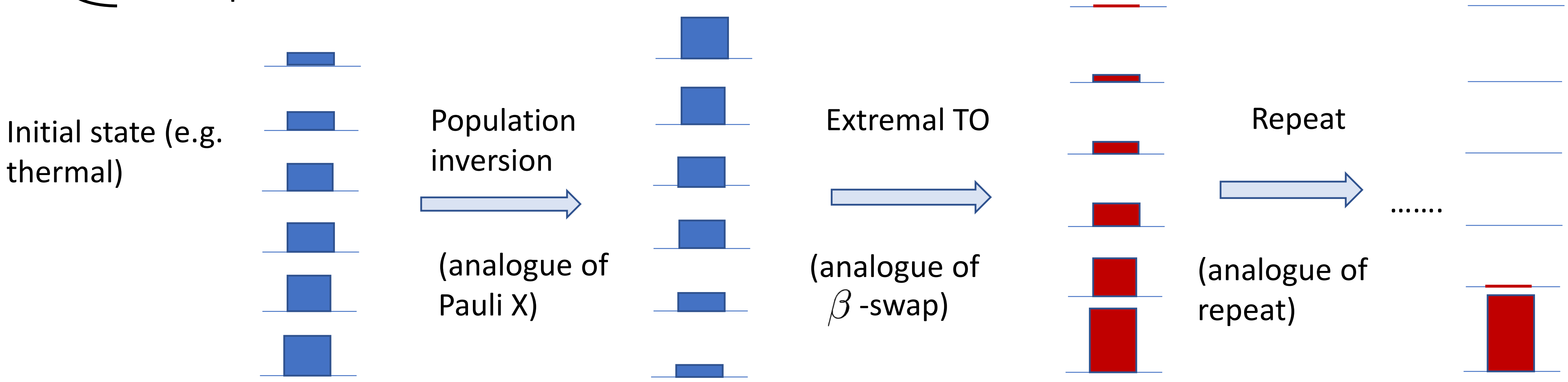
- 3-qubit PPA (Schulman-Mor-Weinstein ‘05)
- SR-$\Gamma_2$ (Rodriguez-Briones et al. ’16)
- HBAC+TO w/ Jaynes-Cummings
- Optimal HBAC+TO



# Optimal cooling of arbitrary qudits

- Steps for qudits (arbitrary dimension): generalization of qubits $$H_S = \sum_{i=0}^{d} E_i |i\rangle\langle i|$$

1. Population inversion: map largest eigenvalue to highest level of Hamiltonian (<u>Max-active state</u>)
2. Extremal thermal operation: maximizes ground state population after applied to a max-active state
3. Repeat



# Optimal cooling of arbitrary qudits: proof ingredients

Key lemma: A “thermal Schur-Horn theorem”

Majorization:

$p'_j \succeq p_i$ $\Longleftrightarrow$ $\exists p(i|j)$ such that $\begin{cases} \sum_j p(i|j) = 1 \\ \sum_j p(i|j) p'_j = p_i \end{cases}$

“Less noisy” “More noisy”

$$\sum_i p(i|j) = 1$$

**Schur-Horn:** Let $\rho$ be a density matrix with eigenvalues $\{p_i(\rho)\}_{i=0}^{d}$ and let $\mathcal{D}(\cdot)$ be a decoherence map. Then

$$p_j(\rho) \succ p_i(\mathcal{D}(U\rho U^\dagger)) \qquad \forall U$$

That is, the eigenvalues of a density matrix $\rho$ **majorize** the probability distribution at the diagonal of $U\rho U^\dagger$

# Optimal cooling of arbitrary qudits: proof ingredients

Key lemma: A "thermal Schur-Horn theorem"

<u>Thermomajorization:</u> finite-temperature equivalent of majorization

$$p'_j \succeq_{\text{th}} p_i \quad \Longleftrightarrow \quad \exists p(i|j) \text{ such that } \begin{cases} \sum_j p(i|j) e^{-\beta E_j} = e^{-\beta E_i} \\ \sum_j p(i|j) p'_j = p_i \end{cases}$$

"Less thermal" "More thermal"

$$\sum_i p(i|j) = 1$$

**<u>Thermal Schur-Horn:</u>** Let $\hat{\rho}$ be a density matrix of a max-active state, (largest eigenvalue in largest energy level etc…), and a Hamiltonian $H_S = \sum_i E_i |i\rangle\langle i|$, and let $\mathcal{D}_{H_S}(\cdot)$ be decoherence in the basis of $H_S$ . Then

$$p(\hat{\rho})_j \succ_{\text{th}} p_i(\mathcal{D}_{H_S}(U\rho U^\dagger)) \quad \forall U$$

That is, the state in which the largest eigenvalues corresponds to the largest energy levels **thermomajorizes** the energy distribution at the diagonal of $U\rho U^\dagger$ .

# Optimal cooling of arbitrary qudits: proof ingredients

- Key lemma: A “thermal Schur-Horn theorem”

  Guarantees that “maximally resourceful/less thermal” state of unitary orbit is the max -active state.

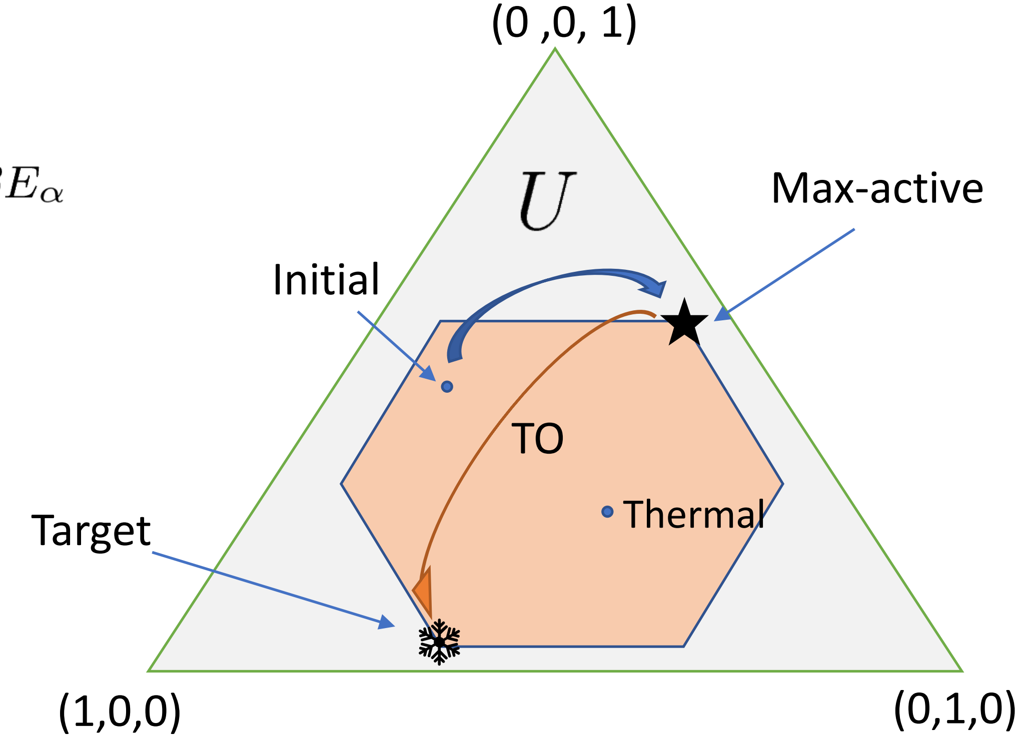
  This way it sets the optimal unitary of the step.

- Then, apply extremal TO that maximizes the population of the ground state.

(Horodecki & Oppenheim ‘13)

$$\begin{pmatrix} \tilde{G}_{\alpha(0)\pi(0)} & \cdots & \tilde{G}_{\alpha(0)\pi(d-1)} \\ \vdots & & \vdots \\ \tilde{G}_{\alpha(d1)\pi(0)} & \cdots & \tilde{G}_{\alpha(d-1)\pi(d-1)} \end{pmatrix} \qquad \tilde{G}_{ij} = \sum_{\alpha} c_{ij}^{\alpha} e^{-\beta E_\alpha}$$

$$p_0^{(k(d-1))} \geq 1 - e^{-k\beta(E_{d-1}-E_0)}\left(1 - p_0^{(0)}\right)$$



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Optimal

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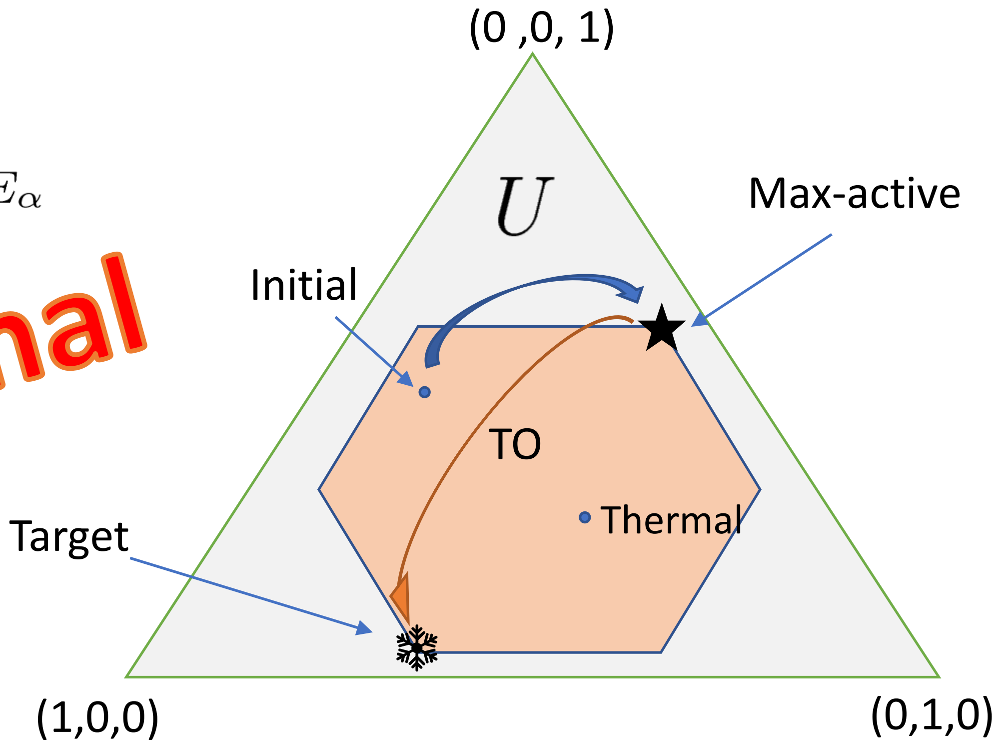

# Conclusions

- We have enhanced the set of operations of HBAC, and found simple algorithms with good performances for single qubits, and for qudits too.
- Why the resource theory picture? It gives a simple set of thermal maps, the “free operations”, whose action we understand well.

$$\beta_S = \begin{pmatrix} 1 - e^{-\beta\Delta} & 1 \\ e^{-\beta\Delta} & 0 \end{pmatrix}$$

- New aspect of RT picture of thermodynamics: use the knowledge of the structure of the theory to optimize over the free operations.
- Other tasks? Similar optimizations?
- Implementations? In which platforms and parameter regimes?

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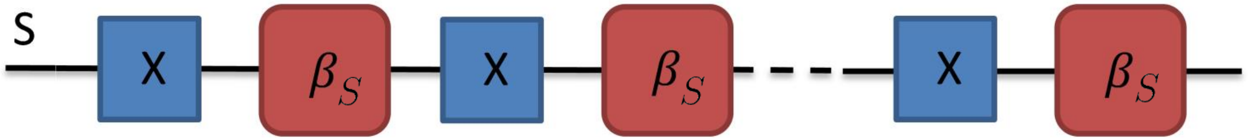
**Thanks!!**

# Cooling qubits




$$p_0^{(k)} = 1 - e^{-k\beta\Delta}(1 - p_0^{(0)})$$

What about the 3rd law of thermodynamics?

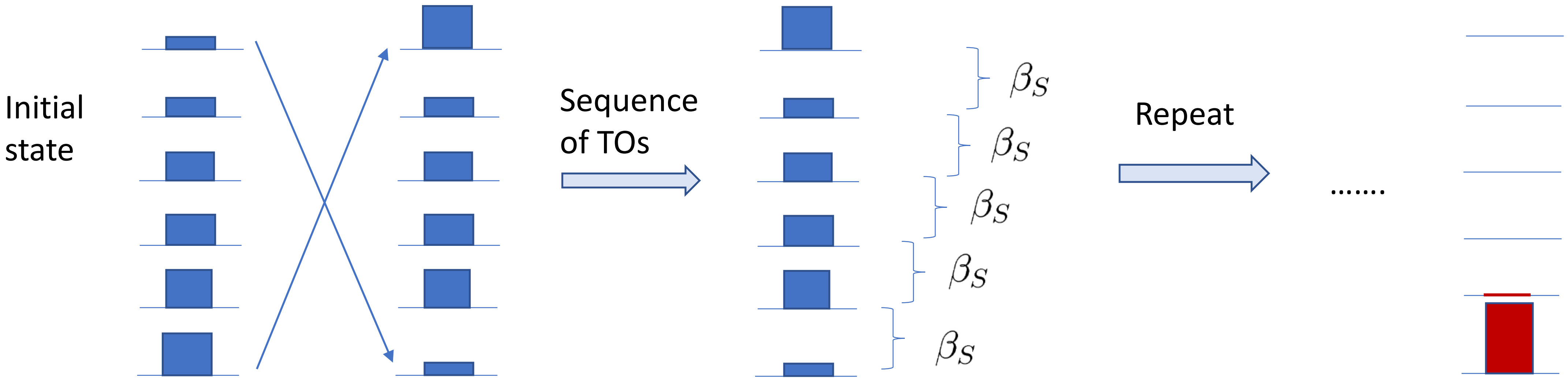
- Perfect cooling requires infinite resources. (Masanes & Oppenheim ’17, Wilming & Gallego ’18)
- Unitary step introduces $w = \pm\Delta$ work (resource) in each round.
- Optimal protocol almost saturates quantitative 3rd law:

$$p_0^{(k)} \leq 1 - C(\beta\Delta)e^{-k\beta\Delta}(1 - p_0^{(0)}) \qquad C(\beta\Delta) < 1$$

(Masanes & Oppenheim ‘17)

# A (simpler) algorithm for qudits

- Another generalization of the optimal one for qubits
- Population inversion between ground state and highest level.
- $\beta$ -swap between subsequent pairs of levels (01)-(12)....(d-1 d).




$$p_0^{(k(d-1))} = 1 - e^{-\beta k(E_{d-1}-E_0)}(1 - p_0^{(0)})$$